1983

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8323291

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Iowa State University PH.D. 1983

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

An investigation of the simulated plant-record (SPR)

balances life analysis model

by

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A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department: Industrial Engineering
Major: Engineering Valuation

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1983

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NOTATION

ASL	direct weighted average service life (area under survivor curve)
b_i	actual annual balance at the end of the ith test year
$\bar{b}$	average of the actual annual balances for a band
bal_v^a	actual balance for vintage v
bal_v^s	simulated balance for vintage v
BG	broad group depreciation procedure
CHGE	annual (constant) change in amount of installations
CI	conformance index calculated by SPR
CM	computed mortality semiactuarial life analysis model
d_i	change in simulated balances between the ith and (i+1)st test year
IC	Iowa curve
INSTS	level installation amount
$INSTS_0$	installation amount for earliest vintage in accounts with nonzero change (CHGE) in installations
IV	index of variation calculated by SPR
m-1	maximum age of the account at the end of the first test year
m	maximum age of the account at the end of the second (middle) test year
m+1	maximum age of the account at the end of the third test year
MLR	maximum life ratio (ratio of maximum life to average life calculated for a survivor curve)
MOG	multiple original group actuarial life analysis model
01-15,5	abbreviation for account in which the O_1 type Iowa survivor curve is used for all vintages and vintage lives decrease by 0.5 each year beginning with a life of 15 for the earliest vintage and ending with a life of 5 for the most recent vintage
O_1-15.5	an O_1 type Iowa survivor curve with an ASL of 15.5

PS_i	the percent surviving on a survivor curve at age i; a superscript of MOG, RR, or IC may be used to describe the type of survivor curve
PS_v	the percent surviving on a survivor curve at the age associated with vintage v; superscripts noted above may be used to describe the type of survivor curve
REI	retirement experience index calculated by SPR
rr	retirement rate calculated for an age interval
RR	retirement rate actuarial life analysis model
SD_b	standard deviation of the actual annual balances
SPR	simulated plant-record life analysis models (used herein to designate the balances model)
SPR-A	depreciation calculated using survivor curves indicated by SPR with the actual vintage balances
SPR-S	depreciation calculated using survivor curves indicated by SPR with simulated vintage balances
SR	survivor rate calculated for an age interval
SSD	sum of squared deviations between simulated and actual annual balances
stub curve	a survivor curve that is not extended to 0% surviving because of a lack of older aged property
VAR_b	variance of the actual annual balances
VG	vintage group depreciation procedure
x	simulated annual balance for the second (middle) test year

CHAPTER I. INTRODUCTION

The recovery of capital, through depreciation expense, is important to all businesses with finite lived property but especially to capital intensive companies such as utilities. Depreciation is often a utility's greatest expense next to fuel, which makes it a major factor in determining the utility's revenue requirements and hence the rates charged to its customers.

The calculation of depreciation requires an estimate of the life characteristics of the property in service. This estimate takes into consideration the present and probable future economic trends, technology, management policies, and other subjective factors. The basis for the estimate, however, is commonly an analysis of the property records. If the utility has kept a record of the age of their property upon retirement, actuarial methods may be used similar to those used by actuaries to estimate human mortality for the determination of life insurance premiums.

Because of the magnitude of the depreciation expense and its use in rate setting, most utilities keep aged records for their major property (e.g., production and transmission systems). Some property, however, is either too numerous or too inexpensive to warrant the difficulty, time, and expense to record the age of each unit upon retirement. Records for this property contain only gross annual amounts, which are mainly installations and retirements. Most utilities have some unaged property (e.g., distribution systems), and small utilities may have all of their property unaged. The Interstate Commerce

Commission (ICC) estimates that approximately 40 percent of the gross investment of the railroads is unaged.[1] Semiactuarial life analysis models are used with these unaged records.

Life estimation is more difficult using unaged records because of the considerable judgment involved in the application of the semi-actuarial models and in the interpretation of the results. The analyst must have a good knowledge of the property and of the life analysis model. A good understanding of semiactuarial models is impeded by a lack of information or published empirical studies. To help fill this void, an investigation of the widely used simulated plant-record (SPR) balances model was undertaken.

The SPR balances model is explained in this chapter along with selected other life analysis models. Since many life analysis models, including SPR, use the generalized survivor curves called the Iowa curves, these curves are discussed following the sections on life analysis models. Because life estimates are often made for the purpose of calculating depreciation, selected depreciation models are described. The chapter ends with a discussion of commonly reported problems with the SPR model and an identification of the problem upon which this study focuses.

Actuarial Life Analysis Models

Actuarial models may be used to analyze property life characteristics when aged property records are available. The retirement rate (RR)

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[1]Hostettler, E. C., Chief, Depreciation Branch, ICC, Washington, D.C., 1983.

and multiple original group (MOG) models each calculate a survivor curve which may be used to estimate the survivor characteristics of the property. The RR model is discussed because of its frequent use in estimating the life of property. The MOG model is discussed because of its relationship to the survivor curves indicated by SPR (see Phase I, Chapter IV). The RR and MOG models are discussed separately below.

Retirement rate model

RR survivor curves are calculated by applying the retirement rate for an age interval to the percent surviving at the beginning of the interval to give the percent surviving at the end of the interval. A retirement rate is the quotient of retirements during the interval and survivors at the beginning of the interval. Unless all installations have the same life characteristics, the curve will vary depending upon the placement or experience band used to calculate the retirement rates. In a placement band analysis, a band of consecutive vintages is followed through each transaction year. The calculated curve represents the actual history of these vintages. In an experience band analysis, the transactions from all vintages that pass through a band of consecutive transaction years are used to calculate the retirement rates.

The RR calculated curve may be matched to a set of generalized curves, such as the Iowa curves (see Iowa Type Survivor Curves in this chapter). The computer simulates a visual match by selecting for each Iowa curve type, a curve that minimizes the sum of differences (positive and negative) between the RR curve and the Iowa curve. These curves are then ranked using a criterion such as the so-called residual measure:

$$\text{residual measure} = \left(\frac{\sum_{i=1}^{n} (PS_i^{RR} - PS_i^{IC})^2}{n} \right)^{1/2} = \left(\frac{SSD}{n}\right)^{1/2}$$

where PS_i^{RR} = percent surviving on RR calculated curve at age i

PS_i^{IC} = percent surviving on Iowa curve at age i.

Further discussion of actuarial models is available (AGA-EEI, 1942; EEI, 1952; Cowles, 1957; Winfrey, 1967; Marston et al., 1953).

Multiple original group model

An MOG survivor curve for an experience band is calculated by dividing each vintage's survivors by its installations and plotting the quotients beginning with those for the recent vintages. If all vintages have the same life characteristics, the RR and MOG curves will be identical.

MOG survivor curves are not necessarily monotonically decreasing as are the RR curves. To see this, assume that all but two vintages passing through the test year have survivors and follow the same dispersion pattern. Of the two exceptions, none of the more recent vintage's installations are surviving while all of the older vintage's installations are surviving. The MOG survivor curve will fall to 0% surviving to represent the more recent vintage and increase to 100% surviving to represent the older vintage.

Semiactuarial Life Analysis Models

Semiactuarial models are used to analyze life characteristics when aged property records are not available. The simulated plant-record (SPR) model is applied to the unaged records of many companies to indicate a generalized survivor curve, usually an Iowa type curve, which purports to represent the property's life characteristics. The computed mortality (CM) model is not currently used to indicate a survivor curve but is used instead to simulate aged data. Two of the turnover models may be combined to indicate a survivor curve although any one of the models by itself can indicate only average life. The SPR, CM, and turnover models are discussed separately below.

Simulated plant-record model

One of three SPR models (balances, annual retirements, period retirements) may be used to indicate a survivor curve. A mathematical representation of the balances and period retirements models may be found in a paper by White (1971).

In the balances model, the Iowa curves are ranked according to each curve's ability to simulate annual balances that are close to the actual annual balances for specified test years. Since an annual balance is the sum of vintage survivors, survivors for each vintage are simulated by distributing the vintage's installations over time according to the mortality characteristics of a survivor curve. Curves are considered sequentially in an effort to match the actual balances. The closeness of the simulated and actual balances is measured by the conformance index (CI) and the index of variation (IV).

For each curve, these indices relate the sum of squared differences (SSD) between simulated and actual balances to the size of the account:

$$CI = \frac{\text{average balance}}{\left(\frac{SSD}{n}\right)^{1/2}}$$

$$IV = 1000/CI = \frac{1000\left(\frac{SSD}{n}\right)^{1/2}}{\text{average balance}}$$

$$\text{where } SSD = \sum_{i=1}^{n} (\text{simulated balance} - \text{actual balance})^2$$

n = number of years in test band.

Since an SSD of zero indicates a perfect match between simulated and actual balances, a low SSD indicates that the curve has generated annual balances that are close to the actual balances. It follows that the highest ranking curves are those with the lowest IVs and the highest CIs. The IV, when divided by 10, approximates the average difference between simulated and actual balances expressed as a percent of the average actual balance.

The maturity of the account is measured by the retirement experience index (REI). For each curve the REI represents, as a percent, the oldest vintage's total retirements as of the end of the most recent test year assuming that the vintage's installations have followed the retirement dispersion pattern of the curve. During the installation period, an REI of 100% indicates that a complete curve has been used in the simulation. An REI less than 100%, say x%, indicates that a survivor curve stubbed at (100-x)% surviving has been used. The higher the REI, the longer the curve stub and, since the Iowa curves become more differentiated with age, the more assurance that a unique curve pattern was used in the simulation.

The SPR period retirements model finds a curve of each type such that the sum of the simulated retirements for the period matches the total actual retirements for the period. These curves are then ranked by least squares (minimum SSD), which White (1971) has shown may be calculated using annual retirements or balances with equivalent results. The annual retirements model simply matches annual retirements under the least squares criterion. The CI, REI, and other indices are used to judge the results.

Computed mortality model

The CM model is used to simulate missing aged mortality data. The aged data may be used to price retirements, make life analyses, and calculate depreciation.

To simulate vintage survivors at the end of a year, survivor rates (or retirement rates) from a dispersion pattern, such as that given by an Iowa curve, are applied to the survivors at the beginning of the year. The ASL of the curve is varied until the sum of vintage survivors matches the actual balance. These vintage survivors are used to simulate the next year's survivors, and so forth. Since there is a curve of each type that can generate vintage survivors that sum to equal the actual balance, the curve type must be specified. The ASL of the curve used in the simulation may be recorded and the time series of recorded ASLs examined for trends which might assist in estimating the property's life characteristics.

In order to begin the CM calculations, beginning of the year vintage survivors are required. For unaged accounts, a beginning age distribution may be simulated using the Indicated Survivors method (NARUC, 1943). This method requires the specification of a curve type and a 1-year test band which, in this case, would be the year prior to the year for which the beginning age distribution is needed.

An advantage of the CM model is that transactions other than additions and retirements, such as transfers and acquisitions, may be aged and incorporated with the simulated survivors in the year of the transaction. A disadvantage is that the curve type must be specified in order to simulate survivors.

Turnover models

Although two of the turnover models may be used to indicate a survivor curve, each of the four models by itself can indicate only average life. The turnover period is the time required to exhaust a specified past balance. One calculation formula cumulates annual retirements backwards until the sum equals a previous balance (EEI, 1952). The number of years between the start of the summation and the date of the previous balance is the turnover period. The period is converted into an estimate of average life using a specified curve type and (constant) growth rate (EEI, 1952).

The turnover period model requires data for a period at least equal to the property's estimated average life. The extensive experience required in order to permit calculations for several periods may prevent the model from being used to indicate trends. To overcome this handicap, the half-cycle ratio model, which requires data for only one-half average life, was developed. This model may be used with the above model to indicate a survivor curve. The model, along with the asymptotic model and its simplified form, the geometric mean model, is based on the ratio of annual retirements to balances.

The simplicity of the turnover models and ease with which they may be applied make them popular. Their use is restricted by assumptions regarding uniform growth rates and homogeneous life characteristics among vintages.

Iowa Type Survivor Curves

The set of Iowa curves was developed to represent the life characteristics of most industrial and utility property. Standard (generalized) curves such as these are used to smooth out irregularities in survivor curves calculated from property records, extend calculated stub curves, and communicate survivor characteristics through the use of widely recognized standard curve shapes.

The development of the Iowa curves began in 1921 with a generalization of 65 retirement frequency curves calculated for industrial and utility property into 13 Pearsonian type curves (Winfrey and Kurtz, 1931). Analysis of 111 more curves resulted in the addition of five curve types (Winfrey, 1967). Couch (1957) added four curve types that comprise the O family of curves, so called because the modal age (age of the maximum ordinate on the retirement frequency curve) is at the origin. The remaining 18 curves were grouped into the L, S, and R families depending upon whether the modal age was left (L) of, coincident (S) with, or right (R) of the curve's average service life (ASL). Since the ASL is calculated by finding the area under the survivor curve (Marston et al., 1953, p. 144), curves of the same type but with different ASL may be generated by varying the area under the curves of a given type (e.g., Winfrey, 1967, p. 84). This process results in an infinite number of curves of the same type. As an example, an R_1 type curve with an ASL of 11.2 (R_1-11.2) is graphed in Figure 2, Chapter V.

The curves in each family were classified according to the modal frequency (height of the maximum ordinate on the retirement frequency curve) (Winfrey, 1967, pp. 70-72) from low mode (e.g., S_0) to high mode (S_6) (see Figure 14, Chapter VI). For the L, S, and R curve families, the mode and retirement dispersion are inversely related: the higher the mode (e.g., S_6), the less the dispersion (i.e., standard deviation). For the O family of curves, the mode and dispersion are directly related. The Iowa curve set has been expanded to 31 curve types by combinations of the original curves to form the "half" curves (e.g., $S_{0.5}$).

To investigate whether the curves were still representative of property mortality characteristics, Russo repeated Winfrey's data collection, testing, and analysis methods. He concluded that "no evidence was found to conclude that the Iowa curve set, as it stands, is not a valid system of standard curves" and that "no evidence was found to conclude that new curve shapes, not now represented in the Iowa curve set, are necessary" (1978, p. 88).

Depreciation Models

Several depreciation models are available for use with the estimated survivor curve (Winfrey, 1942; Lamp, 1980). The models discussed herein use the average service life (ASL) of the group with the straight-line method. If the ASL is the direct weighted average life for the group and if it is used throughout the life of the group, the investment will be recovered by the end of the group (Lamp, 1980).

A property group may consist of a single vintage or a set of vintages. In broad group (BG) depreciation, the same average life is used with all vintages. In vintage group (VG) depreciation, a different average life may be used with each vintage.

The term depreciation can refer to annual depreciation (annual accrual) or accumulated depreciation (depreciation reserve). The annual accrual is the amount of depreciation for the year and may be calculated as follows, assuming zero salvage is realized from retired property:

$$\text{annual accrual} = \frac{\text{average balance}}{\text{ASL}}.$$

For VG depreciation, the ASL calculated for a group of vintages is the reciprocally weighted average of the individual vintage lives using the vintage balances as the weights.

The sum of past annual accruals for the property in service is reflected in the accumulated depreciation account. Assuming that the life characteristics have been estimated accurately, the level of the accumulated depreciation account at time x is given by the following formula, assuming zero salvage:

$$\text{accumulated depreciation} = (\text{end-of-period balance})\ (1 - \frac{E}{ASL})$$

where E = expectancy (the area under the survivor curve to the right of age x divided by the percent surviving at age x).

For the calculation of expectancy, all property included in the balance must be at the same point on the survivor curve, i.e., it must have the same age. Therefore, the accumulated depreciation for an account must be calculated by vintage and the vintage amounts summed.

Critique of Simulated Plant-Record (SPR) Model

The SPR model and index scales were developed and worked satisfactorily when property had relatively stable life characteristics. Today's changing technology has produced accounts which violate the assumption of SPR that the property analyzed by the model has homogeneous life characteristics. This has caused problems in interpreting the results of the model.

Fitch has noticed anomalies when making life studies using SPR (Fitch et al., 1982a). Others who have been involved in numerous life studies using both simulation and actuarial models have stated in personal communication that the SPR model has given inconsistent results such as high mode or origin moded curves for property for which these curves were not believed to be appropriate.

Experimental studies by Shelbourne and Chopp for the AGE-EEI disclosed problems with the model. Shelbourne (1969) had problems with SPR when the growth rate was uniform. Chopp (1971) detected a bias to long-lived property coupled with low mode Iowa curves. He tentatively concluded that the problem with SPR was more in the selection of a proper dispersion pattern than an average life.

Research studies by Erbe, Singh, and Ponder at Iowa State University also exposed problems with the model. Erbe (1971) found that the SPR indicated curves did not match those indicated by the RR model. Ponder (1978) found that SPR rarely selected the curve that was used to generate the data that were being analyzed. Singh (1980) found variation in curve shapes from S_0 to O_3 depending upon the test band being analyzed. All three researchers found that SPR often failed to indicate a unique curve. Because of these problems, they all recommended further investigation of the model.

In studies of real property, Fitch[2] has observed conformance index (CI) values not in Bauhan's scale (see Simulated Plant-Record Model, Chapter II). The indices have also been faulted for not being able to indicate a pattern or trend in actual balances. The SSD criterion, because of the squaring, includes a penalty for large deviations between

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[2]Fitch, W. C., President, Depreciation Programs, Inc., Kalamazoo, Michigan, 1983.

simulated and actual balances. But deviations may arise from frequent accounting lags which do not indicate a change in property life.

Misconceptions regarding the information provided by the retirement experience index (REI) have caused the SPR results to be misinterpreted. A high REI is thought to give more assurance that a unique curve pattern was used in the simulation (see Semiactuarial Life Analysis Models in this chapter). But far less of the curve may be used to significantly influence the simulated balance than is indicated by the currently calculated REI since the installations of early vintages may be insignificant with respect to their effect on the simulated balance.

Several problems with the interpretation of SPR indices and indicated curves have been discussed above. Since little explanation of the model is available to assist in solving these problems, this research was undertaken. Attention was focused primarily on the variance of the indicated curves as specified by the curve mode. The variance has a significant impact on the accumulated depreciation, renewals, and other calculations for which survivor characteristics are required.

CHAPTER II. LITERATURE REVIEW

Semiactuarial models were necessary prior to the 1950s because little actuarial data were kept by utilities. They are still needed today for property for which it is uneconomical or difficult to collect and manage aged property records. The development and subsequent empirical examinations of the simulated plant-record (SPR) model are discussed in this section. The model is popular because of its alleged ability to indicate both a curve shape and ASL using only annual additions and retirements. Other semiactuarial models, such as computed mortality (CM) and the turnover models, are usually used to indicate only average life.

Computed Mortality Model

The CM model simulates unavailable vintage data and thus permits the calculation of vintage group depreciation. Little explanation about CM is available outside of AT&T (1957), where it was developed, and GTE (1974), where its use is facilitated by the computer program COMPMOR (Kaczkowski, 1977). Both AT&T and GTE match annual balances instead of retirements.

Turnover Models

The turnover models include the turnover period model, the Nash model developed by Nash, the half-cycle ratio model developed by Jeynes, and asymptotic model developed by Jeming, and its simplified form, the geometric mean model, also developed by Jeming (NARUC, 1943; EEI, 1952; Marston et al., 1953). AT&T has information on the turnover in SPR computer programs. The observation has also been made that in each family the minima of the SSD curve for each curve type fall on a smooth curve that is concave upwards (EEI, 1952). This assumption has been removed from several SPR programs since it is not always true.

Bauhan (1947) offers advice on the choice of test bands and the adjustment of data for missing installations. Further discussion of data adjustments may be found in the User Documentation of the SIMULATED computer program of the Interstate Commerce Commission (1980).

Bauhan (1947) developed the conformance index (CI) to relate the SSD calculated for each curve to the size of the account (see Semi-actuarial Life Analysis Models, Chapter I). He proposed an arbitrary scale for the CI:

over 75	excellent
50 to 75	good
25 to 50	fair
under 25	poor.

In an effort to eliminate the subjective use of the arbitrary CI scale proposed by Bauhan, R. E. White (1968a,b, 1970) developed a chi-square test to be applied to the actual and simulated balances in item accounts to support or reject the mortality dispersions indicated by SPR. For the development of the test, he statistically modeled SPR balances. In this model, the actual balances are random variables from a multivariate normal distribution whose mean is assumed to be specified by the simulated balances. The chi-square statistic is used to test the hypothesis that the actual balances came from the population specified method, dispersions are sought that match calculated and actual cumulative retirements for several test years. Whiton favored the use of retirements because deviations in retirements expressed as ratios are greater than corresponding balance ratios because of the smaller magnitude of the retirements. According to Whiton, retirements indicate a property's life characteristics better than do plant balances because the latter are influenced by recent additions from which no retirements may have been made, whereas the retirement curves represent only property which has lived its complete life.

Garland (1967, 1968) developed the annual retirements and period retirements methods of the SPR model. In the former, simulated and actual retirements are compared under the least squares criterion. The latter was developed to overcome a shortcoming of the annual retirements method in that dispersion patterns could be chosen that failed to satisfactorily match the total volume of retirements for the period. This problem was solved by adding the requirement that all curves considered under the least squares criterion must first produce simulated retirements equal to the total actual retirements for the period. Garland favored the period retirements method for its ability to reflect life-dispersion pattern shifts. He stated (1967) that considerable testing of the method had demonstrated that it would discover the life-dispersion pattern used to generate the test data. He proposed (1968) several indices including an indicator of account maturity, the cycle index, to accompany the REI. While the REI indicates the total vertical drop of a survivor curve at a given age, it does not indicate the number of years remaining for the curve. This problem is addressed by the cycle index which is the ratio of the age of the oldest vintage to the maximum age of the curve, i.e., the percent of the full life cycle used in the fitting process.

Further discussion of the balances and retirements models is available (Bauhan, 1948; NARUC, 1968; Erbe, 1971; White, 1971; Fitch et al., 1982b).

Studies of SPR

Papers, master's theses, and doctoral dissertations have focused on the SPR model, often in comparison with actuarial models. Shelbourne (1969) compared results of SPR and actuarial analyses of the same data. Data accounts were created using either random variation or expected values. For accounts in which all installations reflected the same life characteristics, the actuarial models indicated the generating pattern as did SPR except when the growth rate was uniform, in which case SPR gave indeterminant results. For the heterogeneous accounts, the actuarial models indicated a dispersion close to the generating pattern and an average life close to the weighted average of the vintage lives.

Chopp (1971) measured the error in the average life indicated by SPR that might be attributed to random variation from a known dispersion pattern. He found uncertainty of about 10% when the data were generated about an average life of 20. He detected a bias toward longer lives and low mode curves. He tentatively concluded that the error inherent in SPR is not so much with the life determination but in the selection of a proper curve shape.

Erbe's master's thesis (1971) includes documentation and use of the following computer programs that he used to compare the SPR balances, SPR period retirements, and RR models:

PGM	generates data (random value and expected value)
TREN	uses rolling and shrinking bands for actuarial analyses
SELEC	fits simple polynomials to retirement ratios for specified bands
SPR	includes balances and period retirements methods
CHISQ	tests the matching of simulated to actual balances
IACURV	generates Iowa curves
LIST	displays account data.

Additional features of the system include pricing the retirements and calculating installations so as to achieve a specified growth or decline in balances. Erbe observed that oftentimes a unique curve was not indicated by SPR. He concluded that the chi-square statistic used with curves indicated by RR or SPR analyses rejected all but the most uncomplicated accounts for low degrees of certainty. No explanation was offered for the variation between curve shapes indicated by SPR and RR analyses.

Ponder (1978) examined the statistical theory of the SPR model and the effectiveness of the balances model in estimating the correct dispersion and service life. She found that SPR seldom selected the curve used to generate the data and had more difficulty when installations were smaller.

Comparisons between RR, SPR balances, and SPR period retirements models were made by Singh (1980). Test accounts were generated using Monte Carlo random sampling. In some cases, SPR was unable to indicate a unique curve. No explanation was offered for the differences in curve shapes indicated by the models. Singh urged that an SPR and actuarial comparative analysis be undertaken using a large number of accounts with known life characteristics.

Fitch, Jensen, and Young reported to the Interstate Commerce Commission (1982a) on an empirical examination of SPR that Fitch and Young began in 1979. Their work was based on the concept of SPR as a model that was developed by Wolf in 1975 (Fitch et al., 1982b). The SPR balances, SPR period retirements, SPR annual retirements, and RR models were applied to data with level installations. The models' results were compared to each other and to the data's known life characteristics with attention focused on the average lives of the indicated curves.

When all installations had the same, or nearly the same, life characteristics, the SPR balances model reflected the common life characteristics except for accounts at stability and for one-year test bands for which the curve was indeterminant as predicted by Bauhan (1947). SPR balances indicated a modest trend in average life when the data contained a marked trend. A mean very close to the average of all vintage lives was indicated when the test band included all of the account history from the first installation to the last retirement. The curve shape indicated by the balances model differed markedly from the shape used to generate the data and from that indicated by the RR model. They recommended that alternatives to the SSD criterion be considered and that the REI be calculated to consider not arbitrarily the oldest vintage but the oldest vintage with significant installations. Further tests were recommended to investigate the effect of variations in the number of installations and in the test bands chosen for analysis.

CHAPTER III. OBJECTIVES

An extensive examination and comparative analysis of the SPR model of life analysis was undertaken in order to better understand the model and explain its idiosyncrasies. Attention was focused on the variance of the survivor curve as indicated by the curve mode. The objectives of this investigation are as follows:

1. To explain, intuitively and mathematically, the results produced by the simulated plant-record (SPR) balances model.
2. To expose idiosyncrasies of the model.
3. To compare the survivor curves resulting from SPR and actuarial analyses of data sets with identical life characteristics.
4. To compare the depreciation calculated from the survivor curves resulting from SPR and actuarial analyses with the book depreciation calculated using the data's known life characteristics.
5. To develop data generation, analysis, and plotting computer programs to assist in fulfilling these objectives.

This study was not intended to exactly delineate the capabilities of the SPR model either by mathematical analysis or by an exhaustive study of cases. Nor was it intended to test hypotheses about SPR generated curves versus known parent populations by statistical methods. This investigation begins an effort to observe how SPR reacts to certain well-characterized and common data problems and to explain the results through a qualitative understanding of SPR logic (and basic mathematical analysis required) so that practicing analysts can use the results from this study in their decision making.

CHAPTER IV. METHODS AND PROCEDURES

The simulated plant-record (SPR) balances model was investigated by examining the results of applying the model to computer generated data sets (accounts) with known life characteristics. The model's results consisted of ranked survivor curves purported to represent an account's life characteristics.

A standard was needed to which the results could be compared. In studies in which data were generated by Monte Carlo random sampling from a fixed retirement dispersion pattern (Erbe, 1971; Ponder, 1978), the parent curve served as a standard to which the SPR indicated curves could be compared using statistical tests such as the chi-square test. This was not a relevant standard in this study since several curves were used to generate the data in a given data set. Two standards were considered:

1. The survivor curves indicated by the RR actuarial life analysis model.
2. The depreciation calculated using the property's known life characteristics.

In Phase I of the study, the first standard was used. The survivor curves ranked highest by SPR were compared to those ranked highest by the RR model applied to the same data. In view of the disparity between the curves indicated by the two models, the SPR indicated curves were further examined in Phase II.

In Phase II, a rationale was sought for the ordering of the curves by SPR or, if appropriate, the lack of ordering. Intuitive and mathematical explanations for the ordering were developed from a study of the model's logic. Attention was focused on the variance of the curves as represented by the curve mode (see Iowa Type Survivor Curves, Chapter I).

In Phase III, the depreciation standard was used to investigate the impact of the differences in survivor curves indicated by the SPR and RR models. The calculated depreciation based on the curves indicated by the two models was compared to the actual (book) depreciation which was calculated by the vintage group procedure using the property's known life characteristics.

The three phases of the study are summarized below:

Phase I	Comparison of SPR and RR results
Phase II	Examination of SPR logic
Phase III	Comparison of depreciation calculations.

Each phase is described separately below after a discussion of the data generated for the study. This chapter concludes with the rationale for the selection of the two standards used in the study.

The Data

A computer program was used to generate both aged and unaged data sets (accounts). Accounts containing aged data were created for use by computer programs that made actuarial analysis calculations. Accounts containing unaged data were created by extracting the gross annual amounts from the aged data accounts. The unaged data accounts were analyzed by an SPR computer program.

The following information was required to generate each account:

Range of vintages

Pattern of installations

Pattern of vintage lives

Iowa type survivor curve.

Each type of information is discussed separately below.

Range of vintages

The number of vintages in each account was kept low (21 or 31) so that the complete retirement history of the account could be analyzed without exceeding the limits of the computer programs used in the study. The earliest vintage in each account was arbitrarily 1940.

Pattern of installations

The level of installations (400 to 2100) was large enough to avoid problems from substantial rounding but low enough to permit investigation of hypotheses using hand calculations. The installations remained level in some accounts in order to minimize the number of variables and assist in the detection and explanation of patterns noticed in the SPR results. In other accounts, installations changed by a fixed amount each year. Since fixed amount changes in installations result in variable percent changes in installations, accounts were also generated in which installations varied by a fixed percent of the previous year's installations.

The amount of change in installations was chosen in consideration of its effect on the width of the periods of growth and decline in annual balances (Figure 1). For example, large increases narrowed the period of declining balances, whereas large decreases narrowed the growth period and narrow periods provide fewer test bands for analysis.

To approximate real world conditions, an account with a step function change in installations was generated in which installations alternated between increasing 50 per year for 3 years and leveling off for 3 years. The installation patterns used in this study are listed in the installations column of Table 1.

Pattern of vintage lives

Like the installations, the vintage lives were kept low (3 to 15) to facilitate hand calculations. Also like the installations, the vintage lives remained level in some accounts. In others, the curve type remained the same and the vintage lives were "nearly level" in that each vintage's life was selected randomly from an interval of width 1.0 or 2.0. Accounts with decreasing vintage lives were generated in order to represent the experience of companies that have increased their retirements in order to be able to install new, technologically superior property. But technological change may have the reverse effect on lives if a company postpones its retirements in anticipation of more advanced replacements in the future. Retirements may also be postponed if depressed economic conditions contract a company's capital budget for replacements. To reflect the condition of postponed retirements, vintage lives were allowed to increase in some data sets. The vintage life patterns used in this study may be found at the top of Table 1.

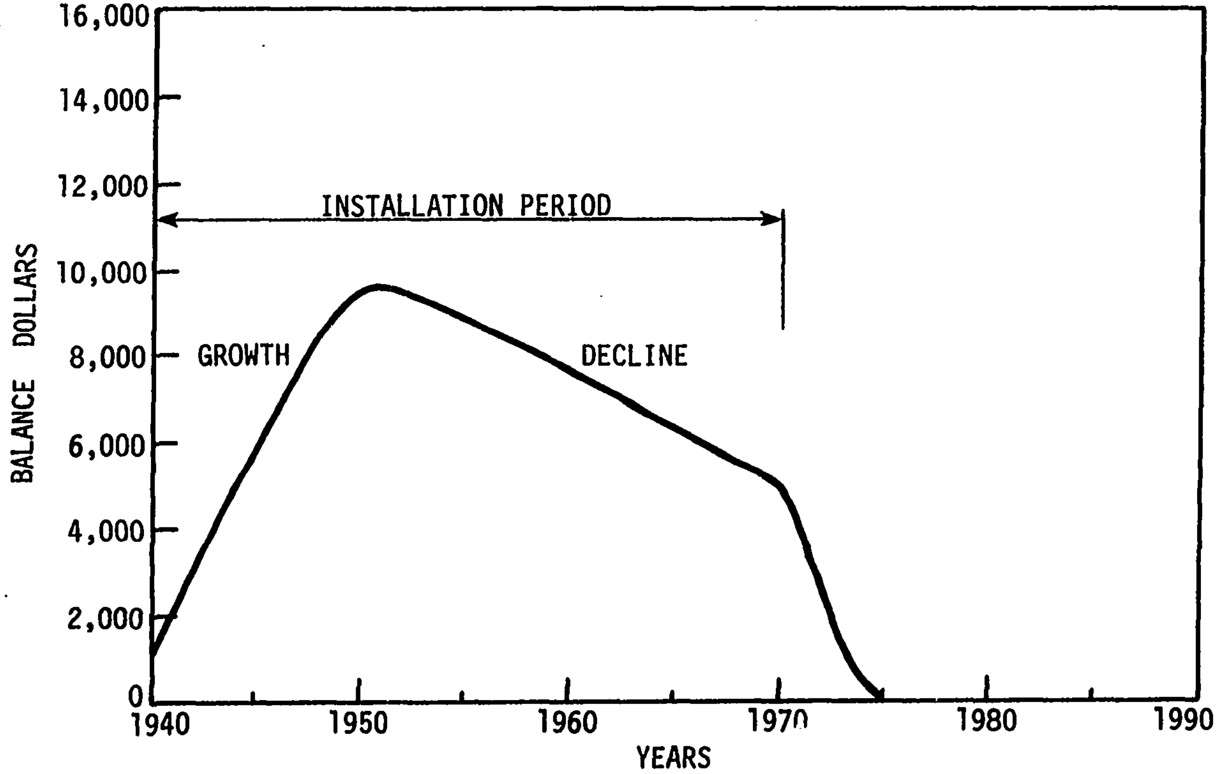

Figure 1. Growth and decline periods of R4-10,4 level installations account

Table 1. Abbreviation codes for the accounts generated for analysis

			Vintage lives			
			Decreasing		Increasing	
Installations	Level	Nearly level	Decrement (-0.5 or -0.2)	Percent (-5% or -2%)	Increment (0.5 or 0.2)	Percent (2%)
Level	01-5[a]	01-5±0.5[b]	01-15,5[c]	01-15,3.2	01-5,15	
(at 1,000)	S2-10	01-5±1.0	S0-15,5		S0-5,15	
		01-15±0.5	S2-15,5		S2-5,15	
		01-15±1.0	S4-15,5		S4-5,15	
			S6-15,5			
			01-10,4	R4-10,5.5	R4-4,10	R4-5,9
			L2-10,4			
			R4-10,4			
			S0-10,4			
			S4-10,4			
			S6-10,4			
Increasing (from 1,000):						
15	01-5			R4-10,5.5		R4-5,9
2%				R4-10,5.5		R4-5,9
2.5%				R4-10,5.5		
Nonuniform				R4-10,5.5		

[a]The O_1-5 Iowa curve was used for each vintage.

[b]The O_1 curve type was used for all vintages. The life for each vintage was selected randomly from the interval (4.5-5.5).

[c]The O_1 curve type was used for all vintages. The vintage lives decreased by 0.5 each year beginning with a 15-year life for the earliest vintage and ending with a 5-year life for the most recent vintage.

Table 1. Continued

Installations	Level	Nearly level	Vintage lives: Decreasing: Decrement (-0.5 or -0.2)	Vintage lives: Decreasing: Percent (-5% or -2%)	Vintage lives: Increasing: Increment (0.5 or 0.2)	Vintage lives: Increasing: Percent (2%)
Decreasing (from 1,000):						
-10	01-5			R4-10,5.5		R4-5,9
-20						R4-5,9
-2%						R4-5,9
-1.5%				R4-10,5.5		R4-5,9

Iowa type survivor curve

The Iowa curve type was held constant in each data set. Seven curve types were used in this study. They were selected by grouping the 22 Iowa type curves according to similarity in the variance of the retirement frequency curves. The eight groups below are ordered by decreasing variance. The curve selected to represent each group is circled. A rationale for the curve selection follows the list.

Group #	Iowa survivor curves			
0	O_3, O_4			
1	O_1, O_2			L_0
2		S_0	R_1	L_1
3		S_1	R_2	L_2
4		S_2	R_3	L_3
5		S_3	R_4	L_4
6		S_4, S_5	R_5	L_5
7		S_6		

The O_1 type survivor curve is a straight line beginning at 100% surviving at age zero and ending at a maximum age equal to twice average life. The simplicity of this curve assisted in the detection of patterns in the data and in the SPR results and in the development of hypotheses. Similar benefits accrued from the symmetry of the curves from the S family. Curves from the R and L families were chosen to represent life characteristics of real world property not represented by the S family and the O_1 curve type and to avoid any bias from the symmetry of retirements of the S family and O_1 curve type. Neither the O_3 nor the O_4 curve types was used since they did not have the benefits of the other curves described above nor was their omission believed to eliminate a large class of property from the study.

Not all curve types were used for each combination of installation and vintage life patterns. One reason for the selectivity was that early findings held regardless of the curve type. Another reason was that this study was intended to expose idiosyncrasies of the model which could be areas for future investigation but not to completely cover each case.

Phase I: Comparison of SPR and RR Results

In Phase I, the survivor curves ranked highest by the simulated plant-record (SPR) balances model were compared to those ranked highest by the retirement rate (RR) model for accounts with level installations. RR survivor curves were calculated for the experience bands that were analyzed by SPR.

For each experience band, the Iowa curve ranked highest by SPR was compared to the Iowa curve which best matched the band's composite survivor curve calculated by the RR model. A computer program was used to match the composite RR curves to the Iowa curves using the residual measure to indicate the closeness of the match (see Actuarial Life Analysis Models, Chapter I).

The experience (test) bands were kept narrow (3 to 5 years) in order to provide several nonoverlapping bands for analysis and to facilitate the hand calculations that were made to investigate hypotheses. The earliest experience years were not used in order to avoid stub curves which were too short to permit a match to a unique Iowa curve. Bands beginning after the end of the installation period were not analyzed since they did not provide early retirements rates to begin the survivor curve.

The disparity between the Iowa curves indicated by the SPR and RR models was investigated using the multiple original group (MOG) life analysis model (see Actuarial Life Analysis Models, Chapter I). The MOG model was chosen because when a 1-year band in an account with level installations is analyzed, the MOG survivor curve for the year encloses area equal to that under an Iowa curve with the highest possible rank by SPR, i.e., a curve that simulates a balance matching the actual balance. Since both curves enclose equal area, one might ask how the shapes of the two curves compare. This question is particularly important since the MOG and RR curves for the same year are similar in shape. Then if differences in SPR and MOG curve shapes can be explained using the equality of the area enclosed by the curves, these explanations might assist in understanding the differences in the survivor curves indicated by the SPR and RR models.

The curve comparisons described above are summarized below for an n-year band:

1. Iowa curves ranked highest by SPR and RR models.
2. Iowa curves ranked highest by SPR vs. the band's n MOG survivor curves.
3. Band's n MOG survivor curves vs. band's composite and n RR calculated survivor curves.

Since the curve comparisons did not involve the testing of hypotheses, statistical tests were not applied.

Phase II: Examination of SPR Logic

In Phase II, intuitive and mathematical explanations were sought for the presence or lack of ordering of survivor curves by the SPR balances model. In analyzing the SPR results for 3-year test bands, attention was focused on the variance of the curves as denoted by the curve mode (see Iowa Type Survivor Curves, Chapter I). In order to minimize the number of variables, each test band lay entirely within a period of growth or decline in annual balances within the installation period, i.e., the period of years during which property was being installed (Figure 1). The two periods are separated by a period of stability in the data sets with level installations and level vintage lives. As in Phase I and for similar reasons, the years at the beginning and end of the experience year range were not analyzed.

The explanations were developed from an examination of the logic of the model and the impact of the least squares curve selection criterion. Patterns in the sum of squared deviations (SSD) and in the conformance index (CI) statistic (see Semiactuarial Life Analysis Models, Chapter I) were explained.

Phase III: Comparison of Depreciation Calculations

In Phase III, the assumption was made that the life characteristics were being estimated in Phase I for use in a depreciation study. This assumption makes it appropriate to investigate the impact of the differences in survivor curves indicated by the SPR and RR models by comparing the depreciation calculated using the different curves.

The depreciation associated with each life analysis model was compared to the actual (book) depreciation. The book depreciation was calculated according to the vintage group (VG) procedure using the vintage curves that were used to generate the data (see Depreciation Models, Chapter I). Since the SPR and RR models indicated only one average account curve instead of separate vintage survivor curves, this single average curve was applied to all vintages according to the broad group (BG) procedure (see Depreciation Models, Chapter I). Because different procedures were used, equality between depreciation levels was not expected.

Both the annual accrual and the accumulated depreciation were calculated. The account balance was used to calculate the annual accrual, whereas vintage balances were needed for the calculation of accumulated depreciation. Vintage balances were contained in the aged data accounts but had to be simulated for the unaged accounts. Vintage balances were simulated using the Iowa curve indicated by the SPR model and the same procedure as that used by SPR for a 1-year test band, i.e., each vintage's installations were multiplied by a percent surviving from the survivor curve. Different from SPR, the vintage balances were not summed and compared to the actual balance.[3]

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[3]An alternate way of simulating vintage survivors that would assure a match to the actual balance would be to age the actual balance independent of the vintages' installations.

It was appropriate to use the actual vintage balances with the curves indicated by the RR model and the simulated balances with the SPR indicated curves. To avoid any bias in depreciation results that may have been introduced by the simulation of the vintage balances, i.e., the choice of simulation method or survivor curve, the SPR indicated curves were also applied to the actual vintage survivors.

For an account with changing life characteristics, a life estimate is usually made periodically at each so-called study year. The estimated survivor curve is used until the next study. In this research, a 5-year period between studies was assumed. The life analysis models were applied to 5-year experience bands ending with each study year. The indicated survivor curve was used to calculate depreciation for the 5 years beginning with the study year and ending with the year prior to the next study year.

The Choice of a Standard

Standard #1: survivor curve indicated by the RR model

The retirement rate (RR) model is commonly used to analyze life characteristics when aged property records are available. The survivor curve calculated from applying the model to experience bands may be used to estimate an average overall curve for use in calculating broad group (BG) depreciation (see Depreciation Models, Chapter I). Likewise, SPR may be applied to experience bands of unaged data to indicate an average curve.

In practice only one of the two models is used depending upon the type of property records available. It is of interest, however, to compare the curves indicated by the two models, where the SPR model uses only the unaged data extracted from the aged data accounts analyzed bv the RR model. The curves indicated by the RR model comprised the standard for comparison in Phase I of this study.

Standard #2: depreciation calculated using the property's known life characteristics

If aged data are either available or simulated (see Semiactuarial Life Analysis Models, Chapter I), vintage group (VG) depreciation may be calculated using estimated vintage survivor curves. Otherwise, BG depreciation is calculated by applying an average survivor curve to all vintages. An average curve is also used by companies who are required by their regulatory agency to use BG depreciation or who find the estimation of vintage curves troublesome. Since both the SPR and RR models are used to estimate average curves, it is of interest to compare the depreciation that would result from the use of the survivor curves indicated by each life analysis model.

The difference in the SPR and RR based depreciation prompts one to ask which is "correct." If the objective is to recover the investment in each vintage by the vintage's final retirement, then VG depreciation using each vintage's known life characteristics is correct. In Phase III of the study, the depreciation calculated using the average curves indicated by the two life analysis models is compared to that calculated by vintage using each vintage's known life characteristics.

CHAPTER V. DISCUSSION OF PHASE I RESULTS

In Phase I, a comparison was made between the Iowa curves ranked highest by the SPR model and those ranked highest by the retirement rate (RR) model applied to the same test band and account. As explained under Phase I, Chapter IV, the disparity in curves indicated by the two models was investigated in a (second) curve comparison between the Iowa curves ranked highest by SPR and the survivor curves calculated by the multiple original group (MOG) model for the same test bands and accounts. In a (third) curve comparison, the MOG curves were found to be similar to the RR curves calculated for the same years. Because of this similarity, the rationale proposed for the differences in SPR and MOG results can be used to explain the differences in SPR and RR results that were noticed in the first curve comparisons.

Each of the three curve comparisons is discussed below. Under each curve comparison, observations are made and supported with explanations and examples. All curves were calculated using accounts which had level installations.

Observations from Curve Comparison #1: Iowa Curves Ranked Highest by SPR and RR Models

Observation 5.1: For most bands in accounts with level installations, the Iowa curve ranked highest by SPR was dissimilar from the Iowa curve best matching the band's composite RR curve.

For example, for the 01-15,5 account, the RR model indicated origin moded (O type) curves, whereas the SPR indicated curves varied from low to high mode as bands from the beginning to end of the experience year range were tested (Table 2 and Figure 2). For the S4-15,5 account, the curves ranked highest by SPR were higher moded than those of the RR model (Table 2 and Figure 3).

Observation 5.2: For some bands in accounts with level installations, SPR failed to indicate a unique survivor curve.

For example, half of the curves listed by SPR in Table 3 had excellent CI and REI values according to Bauhan's scales (see Simulated Plant-Record Model, Chapter II) making it impossible to rank the curves using the scales. None of these highly rated curves resembled the RR indicated origin moded curves, which were the lowest ranked curves by SPR.

All of the curves listed by SPR in Table 4 had CI values that were approximately equal and, according to Bauhan's scale, poorly rated. For the test bands of the "10,4" accounts in Table 5, for which the balances were decreasing, all of the SPR indicated curves had identical indices except those with long tails (e.g., L_0, O_2, O_3, O_4), which were ranked below the other curves.

The RR indicated curve shape, on the other hand, was consistent in Table 5. The shape of the calculated RR curve can be explained by examining the retirement rates that would be calculated for the most recent experience year assuming property is still being installed. The retirement rates affecting the beginning of the curve are calculated using the retirements at the early ages of the most recent vintages. It follows that the RR calculated curve will reflect retirements at the early ages if these vintages have early retirements. For example,

Table 2. Highest ranked curves by RR and SPR models for 01-15,5 and S4-15,5 level installations accounts

	01-15,5[a]		S4-15,5[a]	
	RR	SPR	RR	SPR
1945-49	O_4-27.9[b]	O_2-15.4	S_4-14.6	R_5-14.0
1950-54	O_4-22.3	L_0-13.1	L_4-14.6	R_5-13.7
1955-59	O_3-12.8[c]	R_1-11.2[c]	S_2-11.2[d]	R_5-12.5[d]
1960-64	O_3-8.2	L_4-10.7	L_3-6.9	S_5-8.0
1965-69	--[e]	S_6-9.9	--[e]	S_5-5.5

[a]Account codes explained in Table 1, Chapter IV.

[b]Represents O_4 Iowa curve type with ASL of 27.9.

[c]Graphed in Figure 2.

[d]Graphed in Figure 3.

[e]Survivor curves were not calculated since data were not available to calculate retirement rates for the curves' early age intervals.

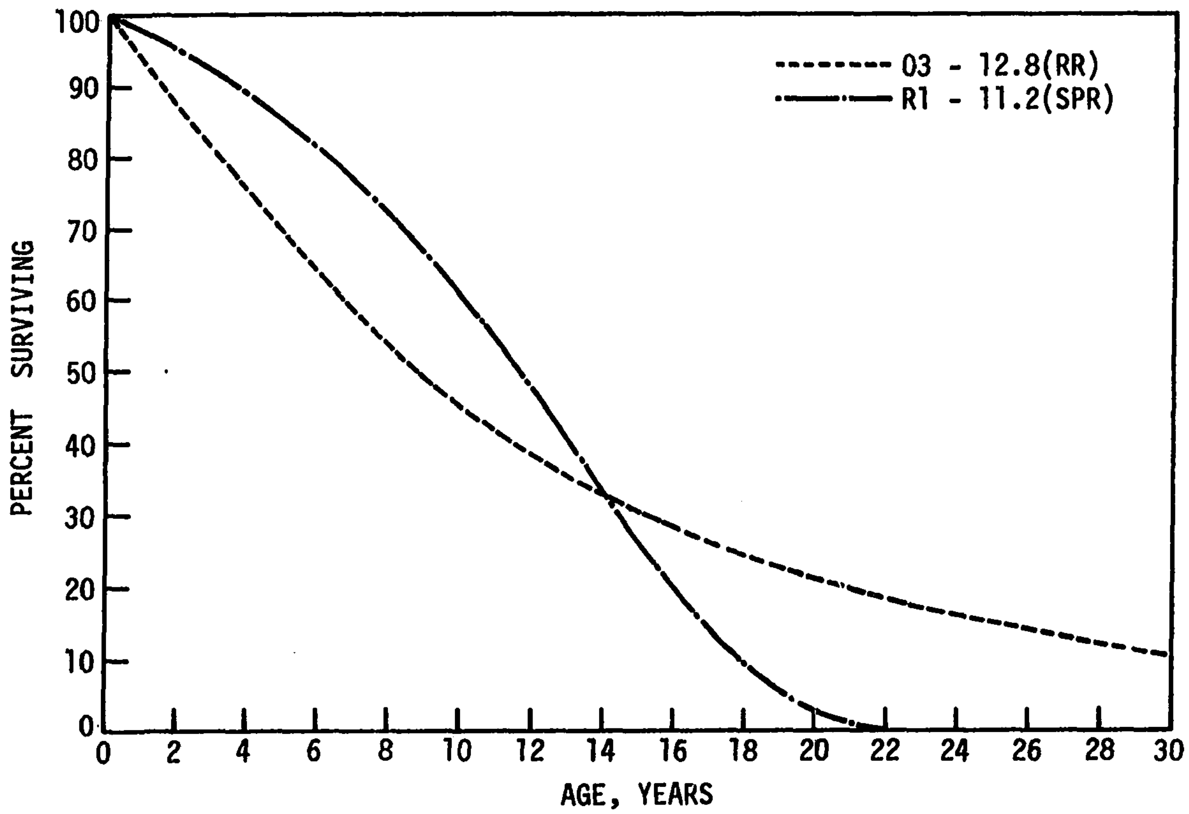

Figure 2. Highest ranked curves by RR and SPR models for 1955-59 band of 01-15,5 level installations account

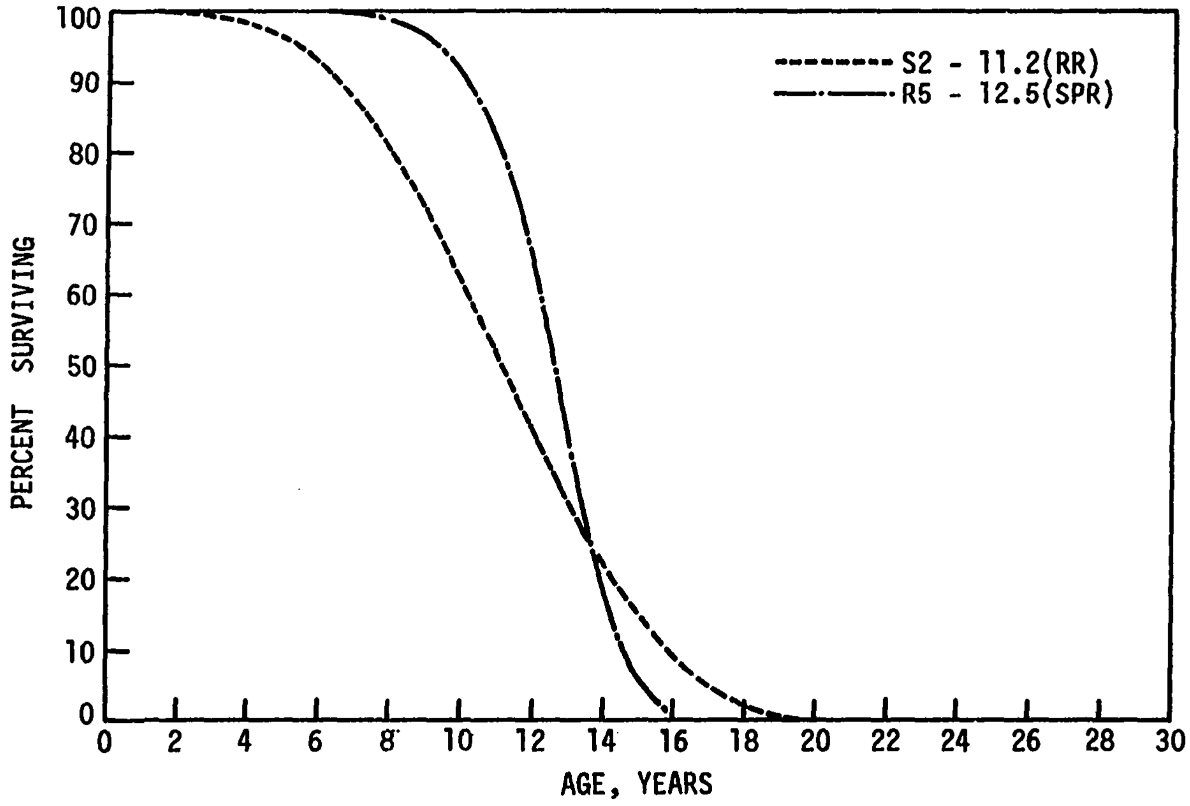

Figure 3. Highest ranked curves by RR and SPR models for 1955-59 band of S4-15,5 level installations account

Table 3. Multiple highly and equally ranked curves by SPR for the 1955-59 band of the 01-15,5 level installations account

Iowa curve type	ASL[a] (years)	CI[b]	REI[c]
R_1	11.255	500	96.15
L_2	11.404	333	93.16
S_0	11.326	250	93.74
S_1	11.178	166	98.19
L_3	11.197	142	97.37
L_1	11.712	111	88.23
R_2	11.118	111	99.81
S_2	11.106	90	99.75
L_4	11.089	83	99.83
S_3	11.077	76	99.99
R_3	11.074	76	100.00
L_5	11.070	71	100.00
S_4	11.068	71	100.00
R_4	11.067	71	100.00
S_6	11.067	71	100.00
R_5	11.067	71	100.00
S_5	11.067	71	100.00
O_1	11.708	62	83.78
L_0	12.387	58	81.46
O_2	13.057	52	79.28
O_3	16.583	34	70.30
O_4	20.867	30	66.53

[a]Average service life (area under the survivor curve).

[b]Conformance index.

[c]Retirement experience index.

Table 4. Multiple equally ranked curves by SPR for the 1955-59 band of the S4-15,5 level installations account

Iowa curve type	ASL (years)	CI	REI
R_5	12.47	14	100.00
S_5	12.51	14	100.00
S_6	12.44	14	100.00
R_4	12.53	14	100.00
S_4	12.45	14	99.99
L_5	12.48	14	99.83
S_3	12.51	13	99.28
R_2	12.54	11	95.65
L_2	13.20	10	85.65
S_1	12.77	11	90.80
L_3	12.74	11	92.35
S_2	12.58	12	96.67
L_4	12.55	13	98.30
R_3	12.47	13	99.68
R_1	12.99	10	85.14
S_0	13.10	10	82.47
L_1	13.81	9	78.43
L_0	15.13	9	70.19
O_1	14.21	8	69.12
O_2	16.02	8	68.00
O_3	21.64	8	60.71
O_4	28.09	8	58.08

Table 5. Highest ranked curves by RR and SPR models for "10,4" level installations accounts

Test band	RR	SPR[a]	RR	SPR[a]
	01-10,4		S0-10,4	
1958-60	0_2-7.8		L_1-7.6	
1961-63	0_2-7.0		L_1-6.8	
1964-66	0_2-6.3		L_1-6.1	
1967-69	0_2-5.5		L_1-5.4	
1970-72	0_2-4.8		L_1-4.6	
	S4-10,4		S6-10,4	
1958-60	L_4-7.5		S_6-8.3	
1961-63	L_4-6.8		S_6-7.6	
1964-66	L_4-6.0		S_6-6.8	
1967-69	L_4-5.3		S_5-6.1	
1970-72	L_4-4.6		S_6-5.4	

[a]All Iowa curves were equally ranked except for lower ranked long-tailed curves (e.g., L_0, 0_2, 0_3, 0_4).

when vintages followed an O_1 type curve which has early retirements, the curve indicated by the RR model (O_2) also had early retirements. The successive retirement rates use earlier vintages which, in an account with decreasing vintage lives, have longer lives and therefore lower retirement rates. This decrease in retirement rates causes the RR curve to level off, i.e., to have a long tail as was true for O_2, L_1, and L_4 curves indicated by the RR model in Table 5.

Observations from Curve Comparison #2: Iowa Curve Ranked Highest by SPR vs. MOG Curves

For each n-year band, an MOG curve was calculated for each year in the band and the n MOG curves compared to the curve indicated by SPR for the band. Before the SPR and MOG results are compared, some general observations about MOG and SPR indicated survivor curves are made.

Observation 5.3: For a band of one year in an account with level installations (INSTS), if a survivor curve simulates a balance equal to the actual annual balance, the following is true:

$$\text{area under survivor curve} = \frac{\text{actual balance}}{\text{INSTS}}.$$

According to the SPR model, an annual balance is simulated by first calculating a balance for each of the n vintages and then summing the n vintage balances:

$$\sum_{v=1}^{n} \text{bal}_v^s = \text{simulated annual balance} \tag{5.1}$$

where bal_v^s = balance simulated for vintage v.

Each vintage balance is simulated by multiplying the vintage's installations by a percent surviving from an Iowa curve:

$$bal_v^s = INSTS \frac{PS_v^{IC}}{100} \tag{5.2}$$

where INSTS = level installation amount

PS_v^{IC} = percent surviving on an Iowa curve corresponding to the vintage's age at the end of the test year.

Substituting (5.2) in (5.1) and using the assumption that the simulated annual balance matches the actual balance:

$$\sum_{v=1}^{n} INSTS \frac{PS_v^{IC}}{100} = \text{actual balance.} \tag{5.3}$$

Dividing by INSTS:

$$\frac{\sum_{v=1}^{n} PS_v^{IC}}{100} = \frac{\text{actual balance}}{INSTS}. \tag{5.4}$$

Because the percents surviving are at 1-year age intervals, the sum of the percents surviving on a survivor curve divided by 100 approximates the area under the curve:

$$\frac{\sum_{v=1}^{n} PS_v}{100} \doteq \text{area under survivor curve.} \tag{5.5}$$

Substituting (5.5) in (5.4):

$$\text{area under survivor curve} = \frac{\text{actual balance}}{\text{INSTS}}.$$

Observation 5.4: For bands of one year in accounts with level installations, SPR cannot indicate a unique curve.

A survivor curve that simulates a balance matching the actual balance encloses the following area (Observation 5.3):

$$\frac{\text{actual balance}}{\text{INSTS}}.$$

Since an infinite number of curves of each type can be generated by varying the area under the curve (i.e., the curve's ASL) (see Iowa Type Survivor Curves, Chapter I), a curve of each type can be found such that the above specified area is enclosed. For example, Figure 4 displays three Iowa curves that each enclosed the required area and therefore simulated a balance that matched the actual balance.

Observation 5.5: For an MOG survivor curve calculated for a 1-year experience band in an account with level installations (INSTS), the following is true:

$$\text{area under survivor curve} = \frac{\text{actual balance}}{\text{INSTS}}.$$

Each percent surviving on an MOG survivor curve is calculated by dividing a vintage's surviving balance by its installations (see Actuarial Life Analysis Models, Chapter I):

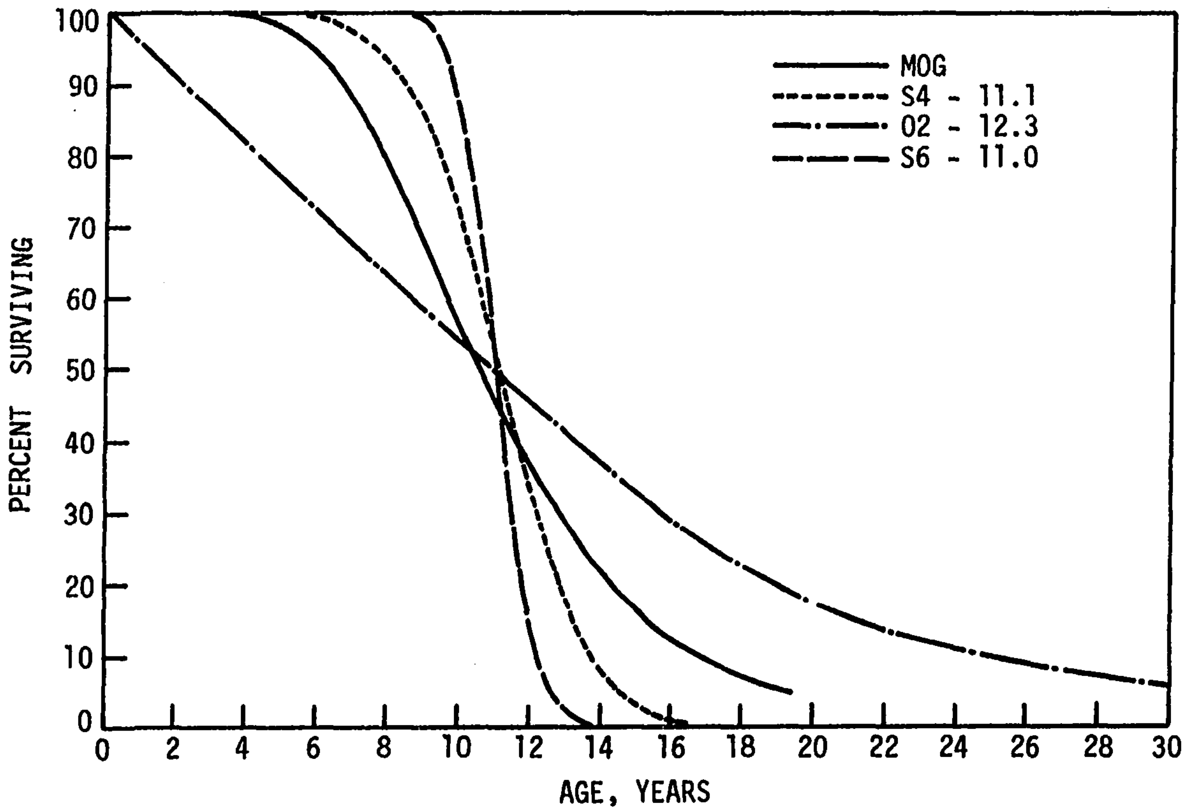

Figure 4. MOG curve and three highly and equally ranked curves by SPR for the year 1959 of S4-15,5 level installations account

$$PS_v^{MOG} = 100 \frac{bal_v^a}{INSTS} \qquad (5.6)$$

where INSTS = level installation amount

PS_v^{MOG} = percent surviving on MOG survivor curve for vintage v

bal_v^a = actual balance for vintage v.

The sum of the percents surviving is expressed as follows:

$$\sum_{v=1}^{n} PS_v^{MOG} = \sum_{v=1}^{n} 100 \frac{bal_v^a}{INSTS} = 100 \frac{\sum_{v=1}^{n} bal_v^a}{INSTS}. \qquad (5.7)$$

Since the percents surviving are at 1-year age intervals, by (5.5):

$$\text{area under survivor curve} = \frac{\sum_{v=1}^{n} bal_v^a}{INSTS}. \qquad (5.8)$$

The sum of the n vintage balances is equal to the actual annual balance:

$$\sum_{v=1}^{n} bal_v^a = \text{actual annual balance}. \qquad (5.9)$$

Substituting (5.9) in (5.8):

$$\text{area under survivor curve} = \frac{\text{actual balance}}{INSTS}.$$

Observation 5.6: For a 1-year test band in an account with level installations, a survivor curve that simulates a balance equal to the actual annual balance encloses area equal to that under the year's MOG survivor curve.

This follows from Observations 5.3 and 5.5. This observation implies that, for an account with level installations, SPR can match an actual balance by finding a survivor curve that matches the area enclosed by, but not necessarily the shape of, the year's MOG survivor curve.

Observation 5.7: For most n-year experience bands, the Iowa curve ranked highest by SPR was dissimilar from the band's n MOG survivor curves. In several bands, the SPR indicated curve was above the MOG curves at the early ages until "cutting through" the set to finish below them.

This "cutting" of the MOG curves by the SPR indicated curve became more pronounced for bands from the beginning to the end of the experience year range of the O1-15,5 account (Figures 5-7). This pattern is also shown in Figure 8 for the S4-15,5 account.

A rationale for this intersection pattern follows. For an n-year band, SPR tries to use a single Iowa curve to match the band's n actual balances or, equivalently, the area under each of the band's n MOG curves (Observation 5.6). If all installations have the same life characteristics, the MOG curves will lie on top of each other so the area under each curve can be matched by an Iowa curve that coincides with the MOG curves. If the life characteristics vary by vintage, the MOG curves will be distinct. The MOG curve for a test year will lie underneath the curve for the previous year (Figure 8). Although the area under a single MOG curve could be matched by an Iowa curve that is a duplicate of the year's MOG curve, the area enclosed by this

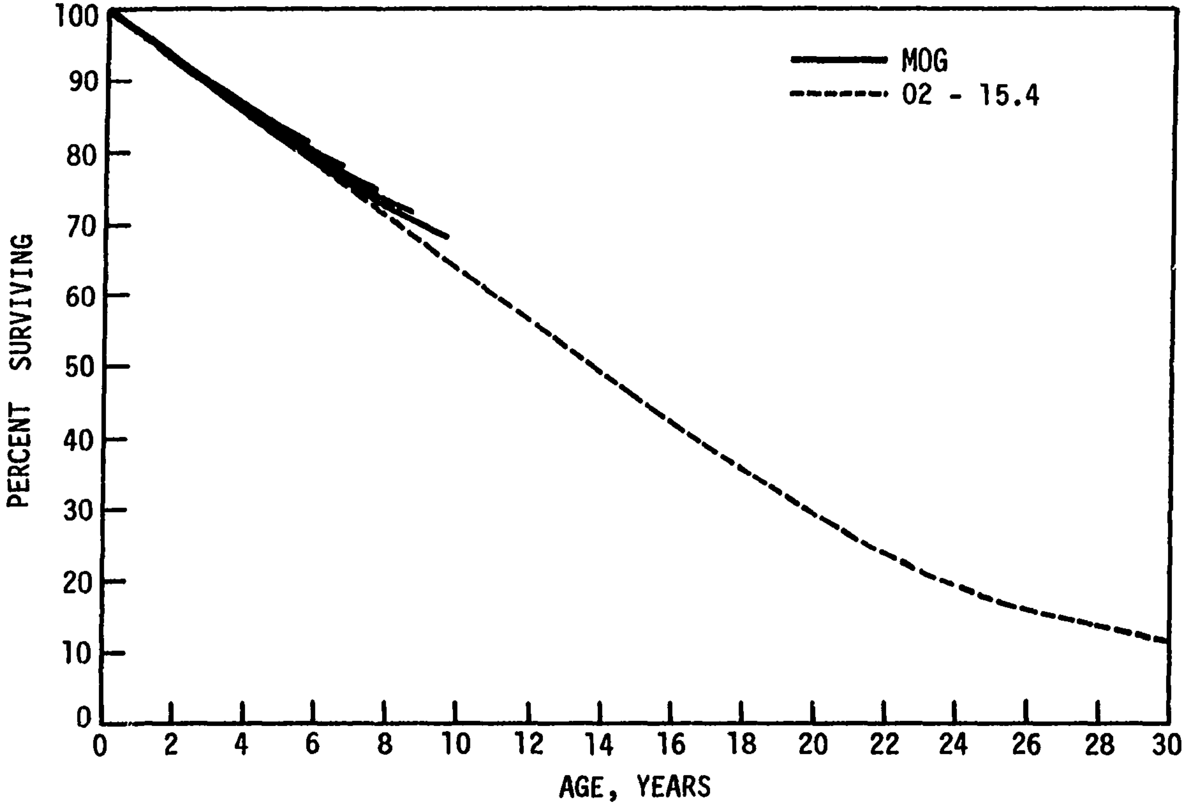

Figure 5. Five MOG curves and highest ranked curve by SPR for 1945-49 band of 01-15,5 level installations account

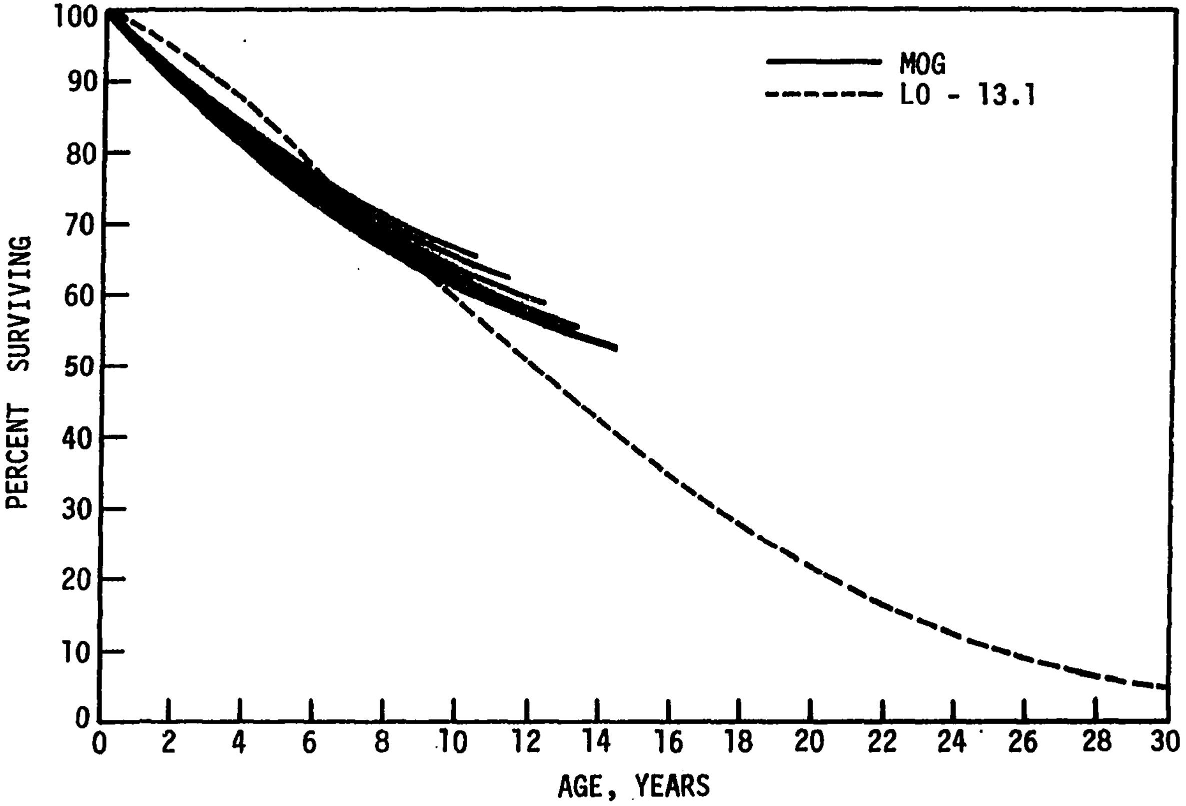

Figure 6. Five MOG curves and highest ranked curve by SPR for 1950-54 band of 01-15,5 level installations account

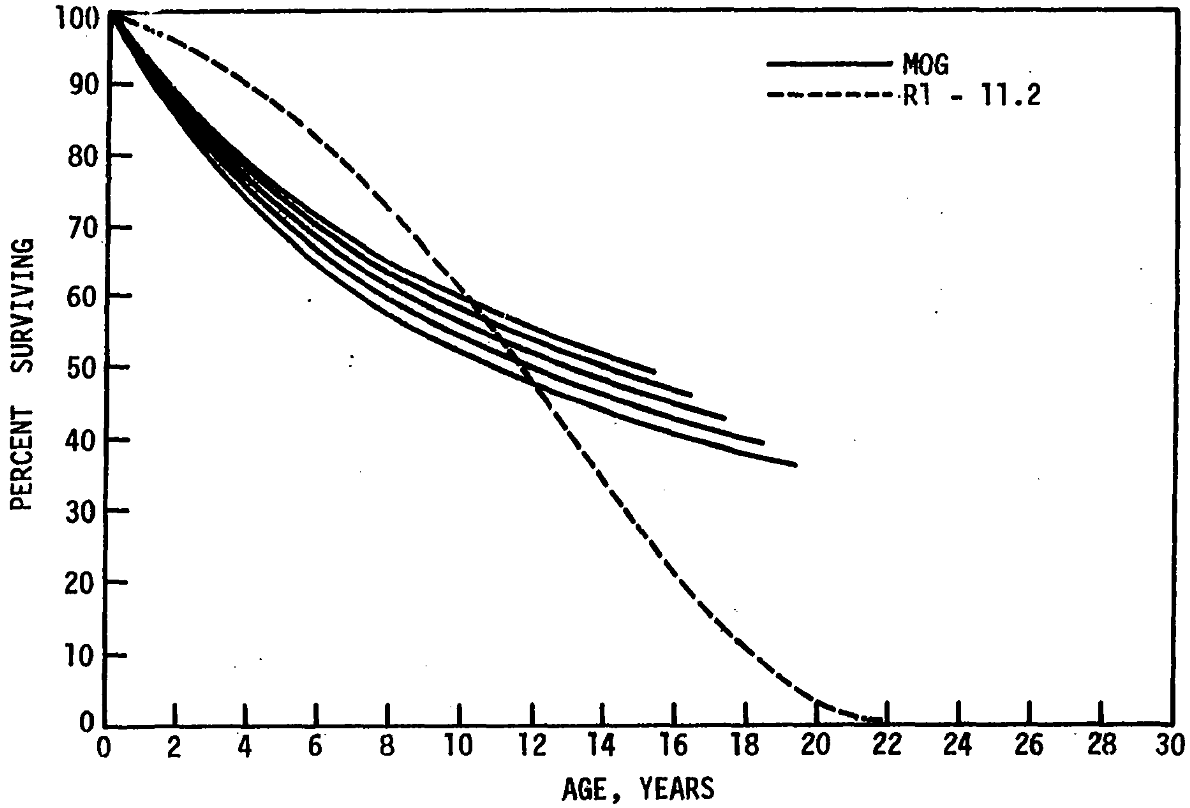

Figure 7. Five MOG curves and highest ranked curve by SPR for 1955-59 band of 01-15,5 level installations account

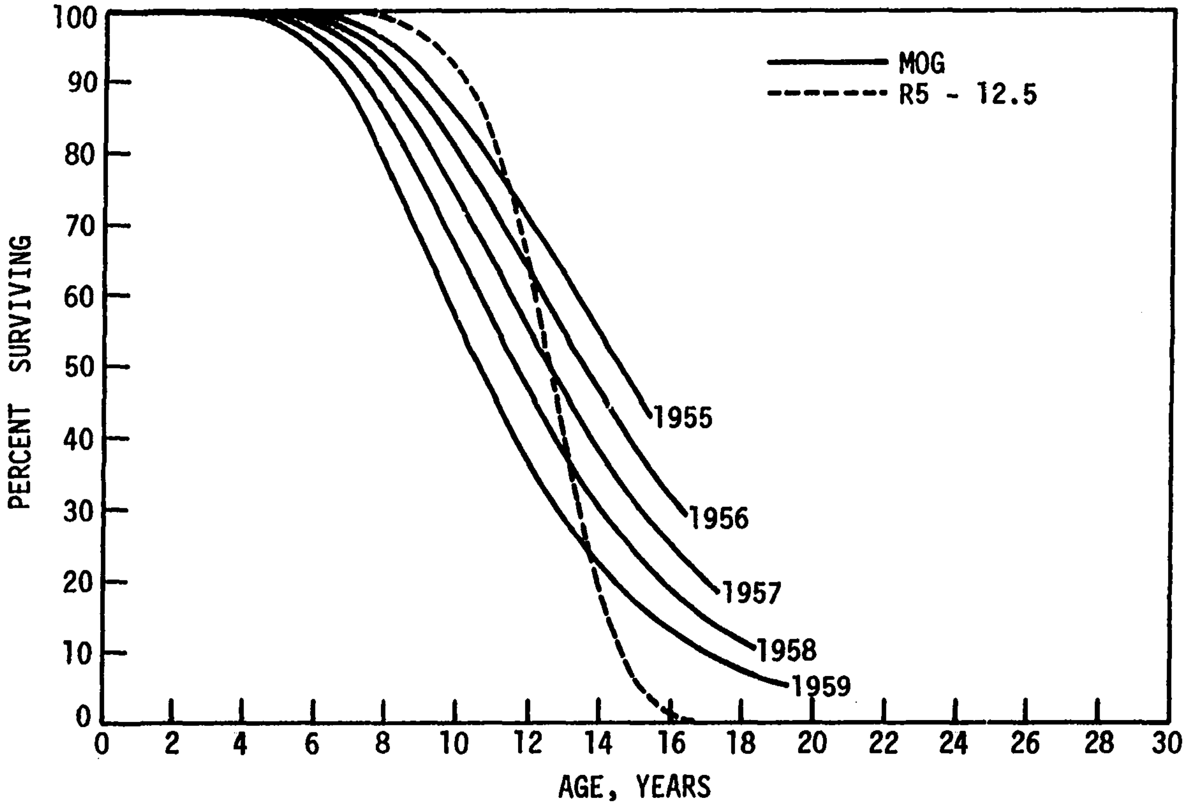

Figure 8. Five MOG curves and highest ranked curve by SPR for 1955-59 band of S4-15.5 level installations account

Iowa curve would exceed that of the MOG curves underneath it and would be less than that of the MOG curves above it. The result would be a large sum of squared deviations (SSD) between simulated balances (represented by the area under the stub Iowa curves) and actual balances (represented by the area under the stub MOG curves) and therefore a low ranking of the Iowa curve by SPR. For example, in Figure 9, the curve ranked highest by SPR is a close match to the middle MOG curve but has a poorly rated CI of 3. Closer matching of the area enclosed by the MOG curves could be achieved by an Iowa curve that balances the area above the MOG curve with area below the MOG curve, i.e., the Iowa curve is both above and below the MOG curve in equal amounts. It follows that the Iowa curve then "cuts through" each of the band's MOG curves.

The implication of this observation will be seen in Curve Comparison #3 in which the MOG curves for a band are shown to be of similar shape to the band's RR curves, so if the SPR indicated curve is not a good match to the MOG curves it will likewise not match the RR curves. This may help explain the disparity in curves noted in Observation 5.1.

Observation 5.8: A high (low) CI for an SPR indicated curve for a band did not necessarily imply that the curve was (was not) similar to the band's MOG curves.

For example, the highest ranked curve by SPR for the three bands in Figures 5-7 had an excellent CI of 500. But the match between the SPR indicated curves and the MOG curves is neither equivalent nor excellent for all three bands. On the contrary, the highest ranked

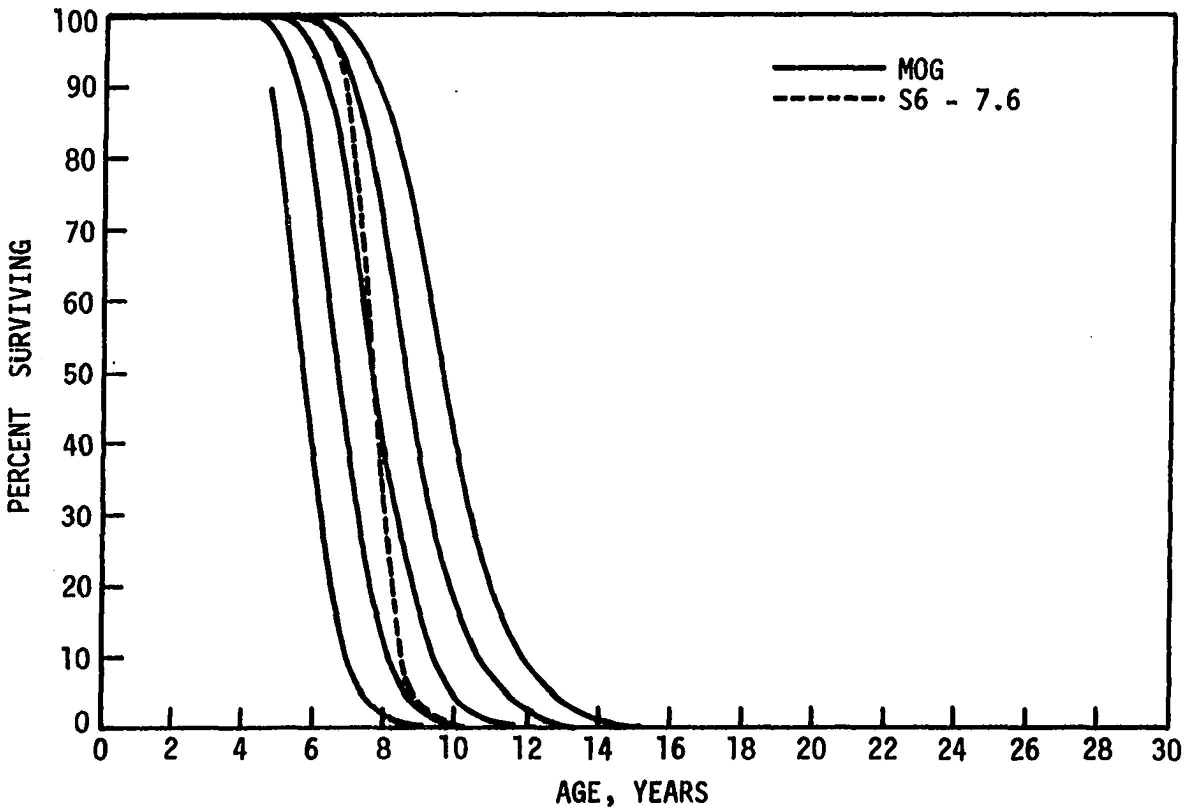

Figure 9. Five MOG curves and highest ranked curve by SPR for 1960-64 band of S6-15,5 level installations account

curve by SPR for the band of the S6-15,5 account in Figure 9 had a poorly rated CI of 13 and yet was a closer match to the MOG curves than were the curves with an excellent CI for the three bands of the 01-15,5 account.

The implication of this observation will be seen in Curve Comparison #3, where it can be concluded that because of the similarity in shape between RR and MOG curves, the CI does not indicate the closeness of the match between the SPR indicated curve and the composite RR curve.

Observation 5.9: For an account with level installations, if a stub survivor curve (not extended to 0% surviving) was used, the simulated balances increased. If a complete curve was used, the simulated balances were equal.

With each successive test year, the range of age intervals for which installations are available is extended by one. If the curve has not reached 0% surviving, the effect is to extend the stub curve and thereby increase the enclosed area and the balance produced by the curve (Observation 5.3). So with each successive test year, the simulated balances increase if a stub curve is used.

If the curve has already reached 0% surviving, the extension does not increase the area under the curve so the simulated balances remain level.

Observation 5.10: For bands after the installation period in accounts with level installations, the simulated balances decreased.

By Observation 5.3, the following holds:

simulated balance = area under survivor curve x INSTS.

The discussion that follows shows that with each successive test year, the area under the survivor curve decreases so the simulated balances decrease.

In the development of Observation 5.9, it was noted that with each successive test year the range of ages for which installations are available is extended by one, which will extend a stub survivor curve and thereby increase the enclosed area. This extension applies also to bands after the installation period. But for these bands, area is lost at the beginning of the curve because the lack of recently installed property means that there are no installations to multiply by the percents surviving at the early ages (beginning) of the survivor curve. The effect is a curve truncated at the front end just as it is stubbed at the tail if older installations are not available. Since survivor curves are monotonically decreasing, the percents surviving and therefore the area lost at the beginning of the curve are greater than the percents surviving and therefore the area added by the extension of a stub curve so the net effect is a decrease in area and therefore in simulated balances. If the survivor curve is complete, the maximum decrease in simulated balances is achieved since there is no curve extension to partially offset the area lost at the beginning of the curve.

Observations from Curve Comparison #3: MOG vs. RR Survivor Curves

Explanations for the disparity between the Iowa curve indicated by SPR and the MOG curves were developed in the previous section. These explanations can assist in understanding the difference between curves indicated by the SPR and RR models because of the similarity (shown below) between RR and MOG curves for accounts with level installations and decreasing vintage lives.

For each n-year experience band, the n single-year RR curves (Figures 10-13) were found to be similar to the band's n MOG curves (Figures 5-8). The similarity between an RR and an MOG survivor curve calculated for the same experience year is discussed below. This section ends by showing that a band's composite RR curve lies close to the band's individual RR curves and hence, by the observations below, to the band's MOG curves. It follows that statements relating SPR indicated curves to MOG curves also relate the SPR indicated curves to the band's composite RR curve.

Observation 5.11: If each vintage has the same life characteristics, the RR and MOG curves will be identical regardless of the pattern of installations.

The RR and MOG curves are shown by mathematical argument and example to be identical for the cases of level and nonlevel installations. The formula below is used to calculate the percent surviving at the end of each age interval on an RR curve as the product of survivor rates for all previous age intervals:

$$PS_n^{RR} = \begin{cases} SR_n PS_{n-1}^{RR} = SR_n SR_{n-1} PS_{n-2}^{RR} = \prod_{i=1}^{n} SR_i \times 100 & \text{for } n \geq 1 \\ 100\% & \text{for } n = 0 \end{cases} \quad (5.10)$$

$$\text{where } SR_n = \frac{\text{survivors at end of age interval } n}{\text{exposures at beginning of age interval } n} \quad \text{for } n \geq 1.$$

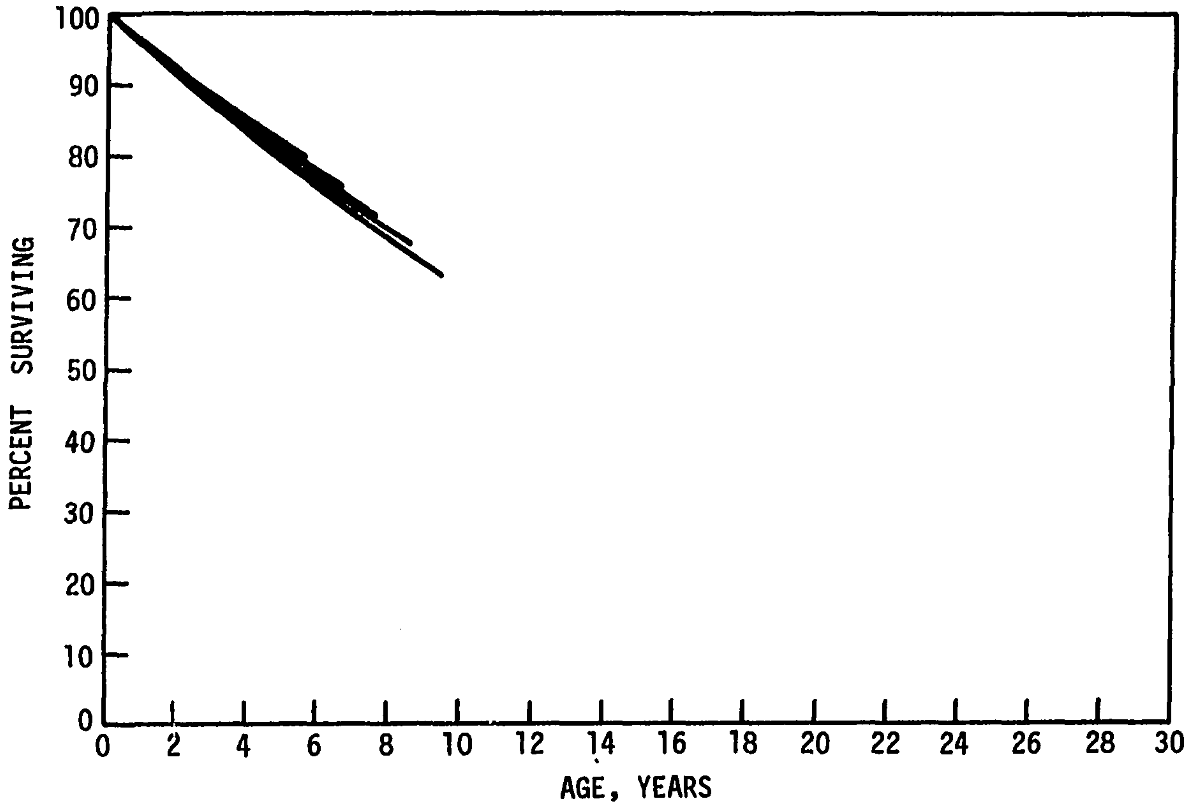

Figure 10. Five single-year RR curves for 1945-49 band of 01-15,5 level installations account

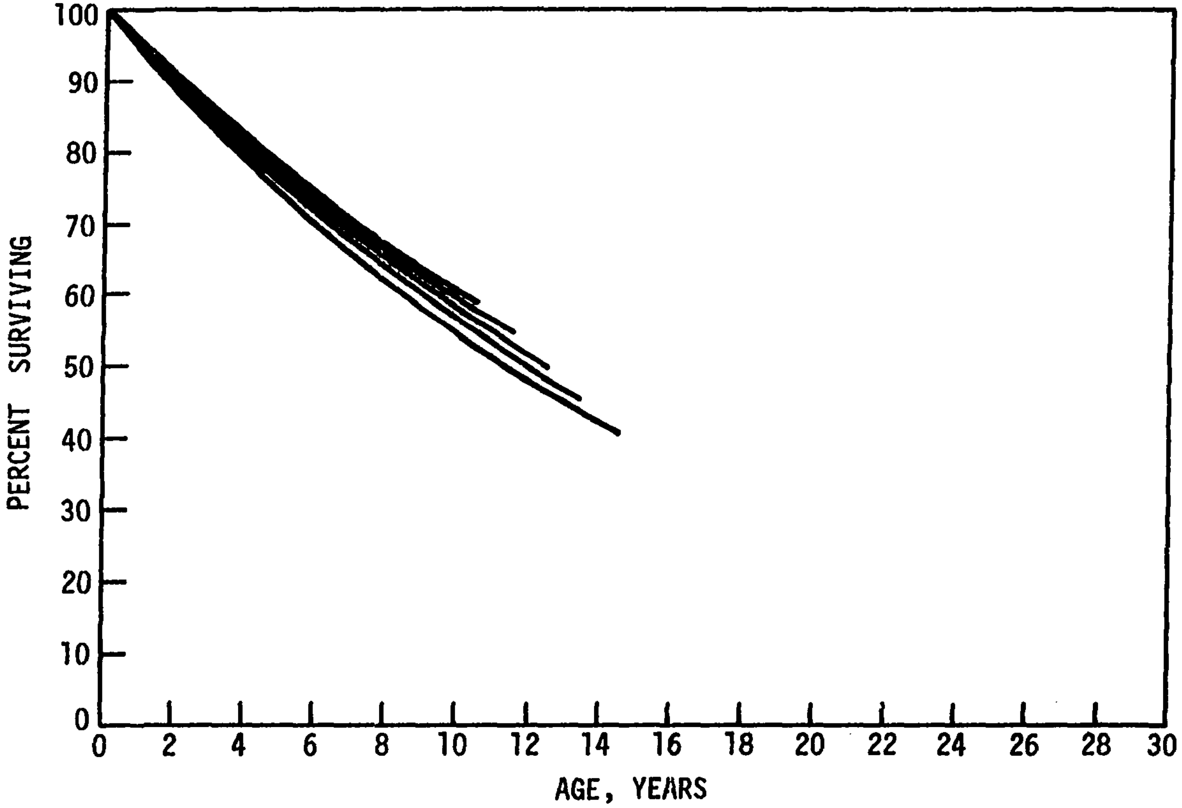

Figure 11. Five single-year RR curves for 1950-54 band of 01-15,5 level installations account

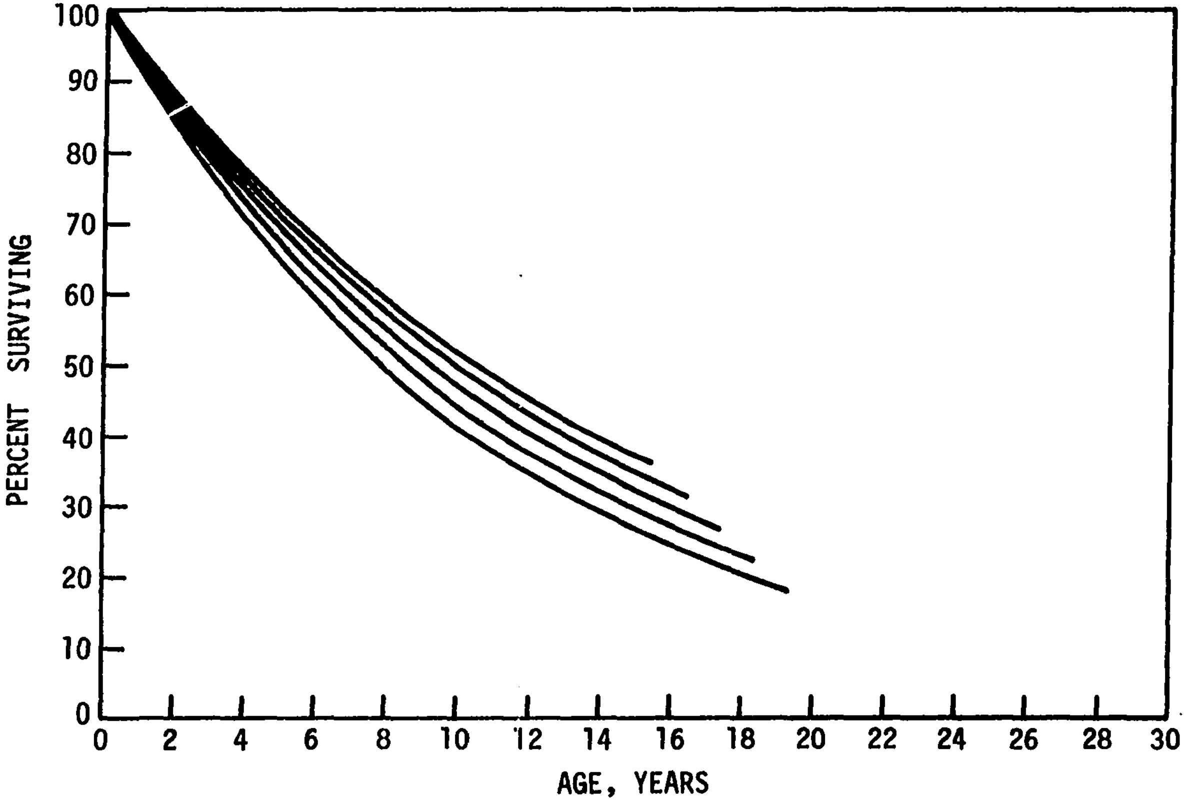

Figure 12. Five single-year RR curves for 1955-59 band of 01-15,5 level installations account

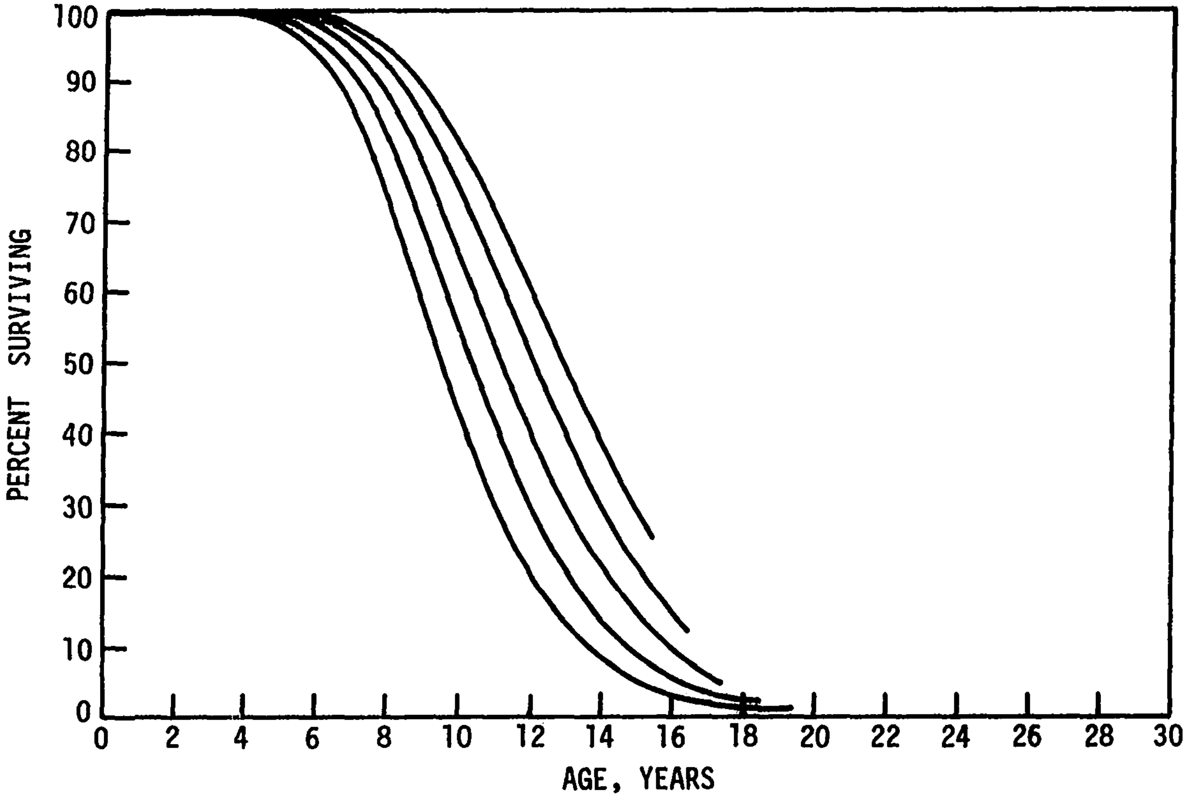

Figure 13. Five single-year RR curves for 1955-59 band of S4-15,5 level installations account

A. Assume installations are level. Consider the account with level installations in Table 6. Since installations are level at 100, this table also represents survivors as a percent of installations. Equation 5.10 is used to calculate PS_3^{RR} (percent surviving at end of age interval 1.5–2.5) for the single experience year 1946:

$$PS_3^{RR} = 0.777 \times 0.818 \times 0.917 \times 100 = 58. \qquad (5.11)$$

To calculate the corresponding point on the MOG survivor curve for 1946, the 1944 vintage's survivors are divided by its installations:

$$PS_3^{MOG} = \frac{58.3}{100.0} \times 100.$$

Or alternatively,

$$PS_3^{MOG} = \frac{58.3}{75.0} \times \frac{75.0}{91.7} \times \frac{91.7}{100.0} \times 100$$

$$= 0.777 \times 0.818 \times 0.917 \times 100. \qquad (5.12)$$

Since (5.11) and (5.12) are equal, it follows that

$$PS_3^{RR} = PS_3^{MOG}.$$

The percents surviving for the RR and MOG curves are equal at each age (Table 7). This follows since the same percent surviving pattern that is repeated in each column appears in each row in Table 6 so that the survivor rates calculated using a row of survivors in an MOG calculation must equal those calculated using two columns of survivors in an RR calculation.

Table 6. 0_1-3 level installations account

Vintage	INSTS	Survivors at end of year 1940	1941	1942	1943	1944	1945	1946	1947
1940	100	91.7	75.0	58.3	41.7	25.0	8.3	0.0	0.0
1941	100		91.7	75.0	58.3	41.7	25.0	8.3	0.0
1942	100			91.7	75.0	58.3	41.7	25.0	8.3
1943	100				91.7	75.0	58.3	41.7	25.0
1944	100					91.7	75.0	58.3	41.7
1945	100						91.7	75.0	58.3
1946	100							91.7	75.0

Table 7. RR and MOG survivor curves for 1946 calculated from 0_1-3 level installations account

n	Age At beg.	Age At end	Survs. at beg.	Rets. during	Survs. at end	Surv. rate	PS at end RR	PS at end MOG
0	0.0	0.0					100	100
1	0.0	0.5	100.0	8.3	91.7	0.917	92	92
2	0.5	1.5	91.7	16.7	75.0	0.818	75	75
3	1.5	2.5	75.0	16.7	58.3	0.777	58[a]	58[a]
4	2.5	3.5	58.3	16.6	41.7	0.715	42	42
5	3.5	4.5	41.7	16.7	25.0	0.600	25	25
6	4.5	5.5	25.0	16.7	8.3	0.332	8	8
7	5.5	6.5	8.3	8.3	0.0	0.000	0	0

[a]Calculation of this percent surviving is discussed herein.

B. Assume installations are not level. A table of survivors for the account can be transformed into a table representing survivors as a percent of installations. This table in percents will duplicate Table 6 since the life characteristics are the same. Therefore, the RR and MOG curves will coincide by part A above.

Since SPR can detect life characteristics if they are the same for each vintage (Observation 6.1), the curve indicated by SPR will be identical to the MOG and the RR curves for accounts in which all property conforms to a single Iowa curve.

Observation 5.12: RR and MOG survivor curves were almost identical for early bands.

This similarity was observed by comparing Figures 5 and 10. The observation follows from Observation 5.11 since survivor rates for the early ages of each vintage are similar even when the life characteristics differ by vintage because many of the Iowa curves have approximately equal rates for the early ages.

Observation 5.13: If the curve type is constant, vintage lives decrease and installations remain level, the RR curve for a single year will lie below the MOG curve for the year.

Consider an account with constant curve type, decreasing vintage lives and level installations (Table 8). The RR and MOG curves calculated for 1943 for this account (Table 9) coincide for the early age intervals since they represent the most recent vintages which in this account have identical life characteristics (Observation 5.11). The RR curve lies below the MOG curve beginning with PS_3 (the percent

Table 8. 0_1 data with decreasing vintage lives and level installations

Vintage	Iowa curve	INSTS	Survivors at end of year 1940	1941	1942	1943
1940	0_1-4	100	93.8	81.3	68.8	56.3
1941	0_1-3	100		91.7	75.0	58.3
1942	0_1-2	100			87.5	62.5
1943	0_1-2	100				87.5

Table 9. RR and MOG survivor curves for 1943 calculated from 0_1 data with decreasing vintage lives and level installations

n	Age At beg. At end	Survs. at beg.	Rets. during	Survs. at end	Surv. rate	PS at end RR	PS at end MOG
0	0.0-0.0					100.0	100.0
1	0.0-0.5	100.0	12.5	87.5	0.875	87.5	87.5
2	0.5-1.5	87.5	25.0	62.5	0.714	62.5	62.5
3	1.5-2.5	75.0	16.7	58.3	0.777	48.5[a]	58.3[a]
4	2.5-3.5	68.8	12.5	56.3	0.818	39.7	56.3

[a]Calculation of this percent surviving is discussed herein.

surviving at age 2.5). To explain this inequality, Equation 5.10 is used to calculate PS_3^{RR} for the single experience year 1943:

$$PS_3^{RR} = 0.777 \times 0.714 \times 0.875 \times 100 = 48.5. \qquad (5.13)$$

To calculate the corresponding point on the MOG survivor curve for 1943, the 1941 vintage's survivors are divided by its installations:

$$PS_3^{MOG} = \frac{58.3}{100.0} \times 100.$$

Or alternatively,

$$PS_3^{MOG} = \frac{58.3}{75.0} \times \frac{75.0}{91.7} \times \frac{91.7}{100.0} \times 100$$
$$= 0.777 \times 0.818 \times 0.917 \times 100. \qquad (5.14)$$

Because each survivor rate in (5.13) is less than or equal to the corresponding rate in (5.14), the following can be concluded:

$$PS_3^{RR} \leq PS_3^{MOG}.$$

This inequality holds for all ages since the survivor rates in (5.14) come from the same vintage and the rates in (5.13) come from vintages which are more recent and therefore have shorter lives (lower survivor rates) since vintage lives are decreasing.

For example, the RR curves in Figures 11-13 lie below the corresponding MOG curves in Figures 6-8. The difference between the curves increases with increasing age.

Observation 5.14: For accounts with level installations and decreasing vintage lives, each band's composite RR curve lay within the boundary determined by the band's n RR curves.

5.6. For a 1-year test band in an account with level installations, a survivor curve that simulates a balance equal to the actual annual balance encloses area equal to that under the year's MOG survivor curve.

5.7. For most n-year experience bands, the Iowa curve ranked highest by SPR was dissimilar from the band's n MOG survivor curves. In several bands, the SPR indicated curve was above the MOG curves at the early ages until "cutting through" the set to finish below them.

5.8. A high (low) CI for an SPR indicated curve for a band did not necessarily imply that the curve was (was not) similar to the band's MOG curves.

5.9. For an account with level installations, if a stub survivor curve (not extended to 0% surviving) was used, the simulated balances increased. If a complete curve was used, the simulated balances were equal.

5.10. For bands after the installation period in accounts with level installations, the simulated balances decreased.

5.11. If each vintage has the same life characteristics, the RR and MOG curves will be identical regardless of the pattern of installations.

5.12. RR and MOG survivor curves were almost identical for early bands.

5.13. If the curve type is constant, vintage lives decrease and installations remain level, the RR curve for a single year will lie below the MOG curve for the year.

5.14. For accounts with level installations and decreasing vintage lives, each band's composite RR curve lay within the boundary determined by the band's n RR curves.

Discussion of observations

For accounts with level installations and decreasing vintage lives, it appears to be inherent in the SPR model that the SPR indicated curve will be higher moded than the composite RR curve. The difference is greatest when the RR curve is origin moded or low moded. The difference is least when balances are decreasing and the RR curve is high moded. The CI is not a good indicator of the similarity between the curves indicated by SPR and the composite RR curve.

If the composite RR curve is considered to be the standard, then based on this study the SPR model cannot be relied upon to meet the standard unless all property has the same life characteristics. The discrepancy is particularly noticeable if the property being analyzed has low mode or origin mode life characteristics.

CHAPTER VI. DISCUSSION OF PHASE II RESULTS

In Phase II, intuitive and mathematical explanations were sought for the presence or lack of ordering of survivor curves by the SPR balances model. As explained in Chapter IV, the results of 3-year test bands were analyzed with attention focused on the variance of the curves as denoted by the curve mode. The accounts of unaged data that were analyzed are described in general in Table 10 and in detail in Table 1, Chapter IV.

All test bands lay within the installation period. General statements about the Iowa curves refer to the R, S, and L type curves. When generalizations include the origin moded (O type) curves, these curves are specifically referenced. The notation used in this chapter is listed on the Notation page.

Observations

Observation 6.1: For the accounts in which all vintages had the same life characteristics, SPR detected the life characteristics, except during stability.

By making small changes in the ASL of each type survivor curve, the SPR model was able to detect the survivor curve that was used to generate the account data. When the vintages had equal installations but not exactly the same life characteristics (see Nearly Level Vintage Lives, Table 1, Chapter IV), SPR indicated the curve type used to generate the account data and an ASL close to the average of the vintages' lives.

Table 10. Classification of accounts generated for analysis

	Vintage lives					
			Decreasing		Increasing	
Installations	Level	Nearly level	Decrement	Percent	Increment	Percent
Level	X	X	X	X	X	X
Increasing:						
Increment	X			X		X
Percent				X		X
Nonuniform				X		
Decreasing:						
Decrement	X			X		X
Percent				X		X

These accounts reflected the assumptions of the SPR model, which performed correctly as expected. Most of the accounts generated for analysis do not conform to the model's assumptions and are used to expose idiosyncrasies in the model.

Observation 6.2: For accounts in which all vintages had equal installations and the same life characteristics, SPR gave indeterminant results when the property was at stability.

Since balances are equal during stability, let each balance equal b. Let the maximum age of the account be m-1, m, m+1 . . . at the end of each year in the test band. If a curve simulates a balance for the first test year that matches the actual balance b, it must enclose area equal to b/INSTS from age 0 to m-1 (Observation 5.3). The curve must be complete in order to produce this balance each year (Observation 5.9). Then all curves that are complete and enclose area equal to b/INSTS by age m-1 will produce balance b each year and have equal, excellent rank according to the SPR model.

In this study, the long-tailed curves (e.g., O_3, O_4) were not able to enclose area equal to b by age m-1 and so ranked lower than the other curves. SPR also gave indeterminant results during the near stability period of the accounts with level installations and nearly the same life characteristics (see Nearly Level Vintage Lives, Table 1, Chapter IV).

Observation 6.3: For accounts with level installations, a complete survivor curve that simulated a balance equal to the actual annual balance had average service life (ASL) equal to the quotient of the actual balance and the level installation amount.

If a curve simulates a balance equal to the actual balance, the following holds (Observation 5.3):

$$\text{area under curve} = \frac{\text{actual balance}}{\text{INSTS}}.$$

If the curve is complete, the enclosed area is the direct weighted average service life (ASL) of the curve (see Iowa Type Survivor Curves, Chapter I) so by substitution:

$$\text{ASL} = \frac{\text{actual balance}}{\text{INSTS}}.$$

Observation 6.4: For accounts with a trend in vintage lives, SPR detected the trend.

Observation 6.5: For accounts with level or increasing installations, the ASL and the mode were inversely related for two SPR indicated curves from the same family.

Consider a high mode and a low mode curve from the same family and with the same ASL as in Figure 14. These two curves have one intersection, before which the higher mode curve is above the lower mode curve. The discussion below shows that if these curves are used to produce the same balance, the ASL's have the following relationship:

$$\text{ASL of higher mode curve} \leq \text{ASL of lower mode curve}.$$

A. Consider an account with level installations. For two curves to produce the same balance, they must enclose equal area (Observation 5.3).

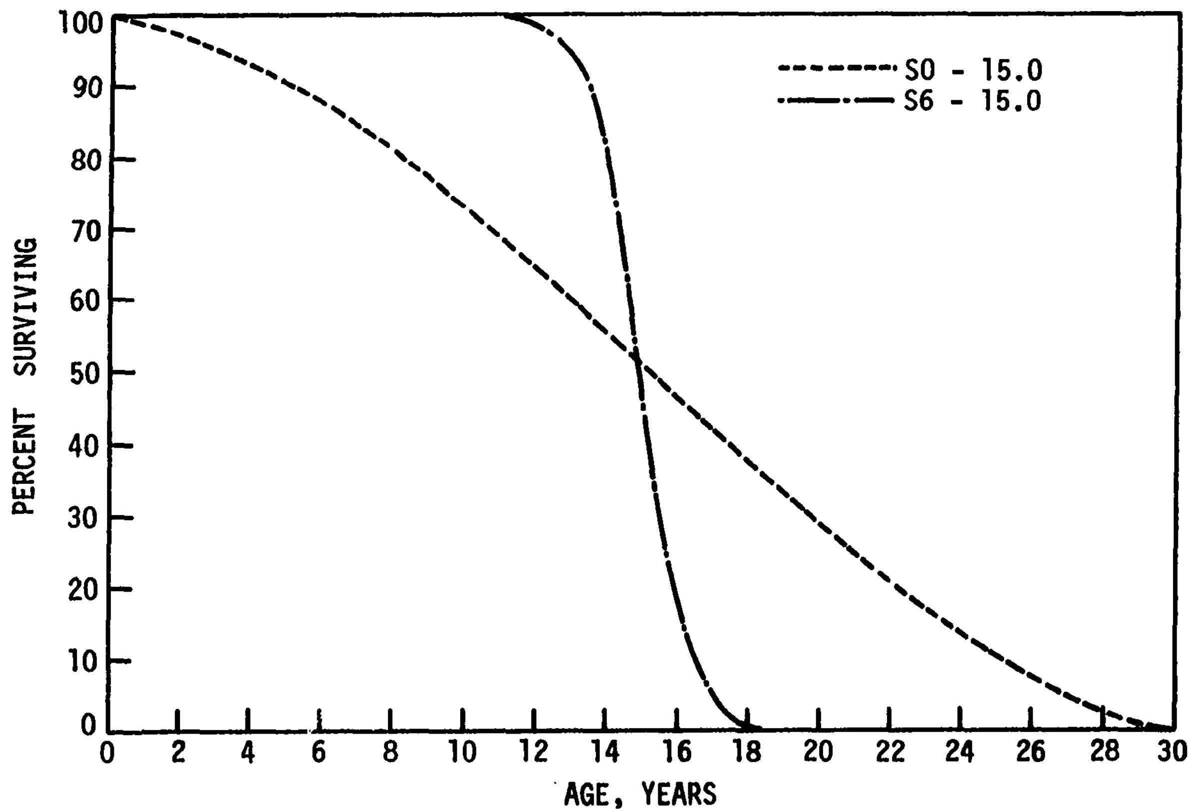

Figure 14. A low mode (S_0) and a high mode (S_6) curve with the same ASL and from the same family

Since the two curves in Figure 14 have the same ASL, they enclose equal area (Observation 6.3) and so will produce the same balance if the complete curves are used. If truncated curves are used because installations are not available for older ages on the curves, the higher mode stub curve will enclose a greater area. The area under the higher mode curve can be reduced to equal that under the lower mode curve by lowering the ASL of the higher mode curve. The ASL of the higher mode curve will then be less than that of the lower mode curve.[4]

B. Consider increasing installations. In this case, two curves that enclose equal area will not produce the same balance since a survivor curve is weighted more heavily at the early ages when installations are increasing and at the early ages a higher mode curve is above a lower mode curve (Figure 14). To offset the effect of the weighting, the front part of the higher mode curve can be lowered by decreasing the curve's ASL. The ASL of the higher mode curve will then be less than that of the lower mode curve.[4]

This observation will be used to conclude that SPR in some cases has a preference for curves with the least ASL of the indicated curves.

In the following observations, the maximum age of the account is denoted by: m-1, at the end of first test year; m, at the end of the second test year; m+1, at the end of the third test year.

Observation 6.6: For 3-year bands in accounts with level installations (INSTS), the following held for the survivor curves that produced

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[4]In this study, the ASL and mode were inversely related even when the optimal pattern of simulated balances (Equation 6.4) indicated that the higher mode curve was seeking a balance greater than that of the lower mode curve.

simulated balances matching the actual balances:

a. The area enclosed by the survivor curve stubbed at age m multiplied by INSTS equaled the actual balance of the second (middle) test year.

b. The height (percent surviving) of the survivor curve at ages m and m+1 multiplied by INSTS equaled the increase in actual balances between the first and second and between the second and third test years, respectively.

a. Follows from Observation 5.3.

b. Follows from Equation 5.2.

Observation 6.7: For 3-year bands in accounts with level installations, steeply (gradually) increasing actual balances (b_1, b_2, b_3) were matched best by a survivor curve that enclosed area equal to b_1/INSTS by age m-1 and had a high (low) height at ages m and m+1.

If a survivor curve simulates a balance that matches the actual balance b_1 in an account with level installations, the curve must enclose area equal to b_1/INSTS by the maximum age of the account, m-1 (Observation 5.3). For the following discussion of increases in balances, consider a high mode and a low mode curve from the same family as in Figure 14. It is not necessary that the curves have the same ASL.

A. Consider steeply increasing balances. When age m is before the intersection of two curves from the same family, the higher mode curve will be above the lower mode curve and so will produce more steeply increasing balances (Figure 14 and Observation 6.6b). The result is a preference by SPR for high mode curves (Table 11). In this

Table 11. SPR results for 1946-48 band of steeply increasing balances of R4-10,4 level installations account

Iowa curve type	ASL (years)	CI	REI
R_4	9.808	1,000	24.56
S_3	10.252	1,000	26.32
L_3	11.399	1,000	23.36
L_4	9.813	500	29.30
S_2	11.830	333	20.43
L_2	14.263	200	17.72
R_3	11.704	200	16.94
S_1	15.062	166	15.42
S_4	9.230	166	34.93
R_5	8.854	125	37.80
L_1	20.668	125	13.04
L_5	9.043	125	38.56
S_0	22.411	111	11.62
R_2	17.058	100	11.21
L_0	35.220	100	10.30
R_1	30.330	90	8.96
O_2	59.234	83	8.63
O_1	52.704	83	8.56
O_3	87.081	83	8.85
O_4	118.415	83	9.12
S_5	8.742	83	44.13
S_6	8.538	58	52.89

table, however, the highest mode curves (e.g., R_5, L_5, S_5, S_6) are not ranked highest since they produced balances with too steep an increase. When age m is after the intersection of two curves from the same family, the lower mode curve will be above the higher mode curve and so will produce more steeply increasing balances and be preferred by SPR (Table 12). As the mode increases, the height of the survivor curve decreases which decreases the slope of the simulated balances (Observation 6.6b). The result is a lower ranking by SPR as the mode increases in each family (Table 12).

B. Consider gradually increasing balances. When age m is before the intersection of two curves from the same family, the curve with the lower mode will produce more gradually increasing balances (Figure 14 and Observation 6.6b). After the intersection, the higher mode curve will produce more gradually increasing balances and be preferred by SPR (Table 13). In Tables 13 and 14, the highest mode curves (e.g., S_5, S_6) are not ranked highest since they produced balances with too gradual an increase. In Table 14, not just the high mode curves but over half of the curves simulated balances with too gradual an increase. The simulated balances were level for half of the curves since age m was after each curve's maximum age so the complete curve was used in the simulation (Observation 5.9). Because there were lower mode curves that were not yet complete and so could produce increasing balances, they were the highest ranked curves. In Table 15, only the long-tailed O_3 and O_4 curves were not complete and so able to produce increasing balances and be the highest ranked curves. Most of the other curves had the same CI since they were

Table 12. SPR results for 1951-53 band of steeply increasing balances of 01-10,4 level installations account

Iowa curve type	ASL (years)	CI	REI
L_0	9.697	500	74.64
O_1	9.173	333	74.09
L_1	9.048	250	81.93
O_2	10.274	250	72.45
S_0	8.701	142	85.88
R_1	8.618	100	88.95
L_2	8.736	100	88.22
O_3	13.349	90	64.99
O_4	17.012	76	62.04
S_1	8.519	66	93.40
L_3	8.526	62	93.93
R_2	8.447	52	97.14
S_2	8.430	50	97.96
L_4	8.407	45	98.96
S_3	8.394	43	99.67
R_3	8.391	41	99.94
L_5	8.384	41	99.93
S_4	8.381	41	100.00
R_4	8.380	41	100.00
R_5	8.379	41	100.00
S_5	8.380	41	100.00
S_6	8.380	41	100.00

Table 13. SPR results for 1949-51 band of gradually increasing balances of R4-10,5.5 level installations account

Iowa curve type	ASL (years)	CI	REI
S_4	9.632	500	87.62
R_4	9.676	250	84.11
L_5	9.623	200	89.41
L_4	9.918	200	78.96
R_5	9.506	111	96.72
S_3	9.944	100	74.41
S_5	9.492	100	97.01
L_3	10.636	66	66.18
R_3	10.044	66	68.76
S_6	9.424	58	99.91
S_2	10.442	55	63.06
L_2	11.735	41	54.85
S_1	11.382	38	51.85
R_2	10.985	37	51.95
L_1	13.672	31	44.05
S_0	13.172	29	41.90
R_1	13.151	27	38.88
L_0	17.146	26	36.48
O_2	20.450	24	32.19
O_1	18.190	24	32.11
C_3	29.428	24	31.58
O_4	39.565	23	31.40

Table 14. SPR results for 1954-56 band of gradually increasing balances of 01-10,4 level installations account

Iowa curve type	ASL (years)	CI	REI
S_0	8.871	1,000	98.17
L_2	8.936	1,000	96.00
R_1	8.859	500	99.05
L_3	8.861	500	98.85
S_1	8.846	333	99.71
L_1	9.063	250	92.50
S_2	8.838	250	99.99
R_2	8.838	250	100.00
L_4	8.838	250	99.97
S_3	8.837	250	100.00
L_5	8.837	250	100.00
S_6	8.836	250	100.00
R_5	8.837	250	100.00
S_4	8.837	250	100.00
S_5	8.837	250	100.00
R_4	8.837	250	100.00
R_3	8.837	250	100.00
O_1	8.952	200	92.66
L_0	9.392	111	87.00
O_2	9.819	90	84.47
O_3	11.921	52	75.59
O_4	14.565	43	71.29

Table 15. SPR results for 1954-56 band of gradually increasing balances of R4-4,10 level installations account

Iowa curve type	ASL (years)	CI	REI
O_4	6.820	333	86.53
O_3	6.338	100	91.67
O_2	5.951	52	97.90
L_0	5.897	47	99.43
L_1	5.882	45	99.98
L_2	5.881	43	100.00
L_5	5.881	43	100.00
R_4	5.880	43	100.00
S_3	5.881	43	100.00
R_3	5.881	43	100.00
S_4	5.881	43	100.00
S_0	5.881	43	100.00
R_5	5.881	43	100.00
R_1	5.883	43	100.00
O_1	5.883	43	100.00
S_5	5.881	43	100.00
R_2	5.881	43	100.00
L_3	5.881	43	100.00
S_2	5.881	43	100.00
S_1	5.881	43	100.00
L_4	5.881	43	100.00
S_6	5.877	43	100.00

complete and so produced level simulated balances equal to ASL x INSTS (Observation 6.3).

Observation 6.8: Three-year bands of level or decreasing balances in accounts with level installations were matched best by level simulated balances equal to $\bar{b}$, the average of the band's actual balances.

Although decreasing actual balances cannot be matched by SPR if installations are level or increasing because simulated balances cannot decrease (Observation 5.9), the pattern of nondecreasing simulated balances which minimizes the SSD can be determined. In part A, the optimum pattern for complete survivor curves is found to be level balances at $\bar{b}$. In part B, the optimal balances simulated by a curve of unspecified length are found. In part C, the SSD for the optimal patterns found in parts A and B are compared and the SSD is found to be minimal for level simulated balances equal to $\bar{b}$.

The following notation (see Notation page) is used:

b_i actual balance at the end of the ith test year.

$\bar{b}$ average of the actual balances for the band.

x simulated balance for the second (middle) test year.

d_1 change in simulated balances between first and second test years.

d_2 change in simulated balances between second and third test years.

SSD sum of squared deviations between simulated and actual balances.

Using the above notation, simulated balances for the three test years are as follows, where $d_1 \geq 0$ and $d_2 \geq 0$ since simulated balances cannot decrease (Observation 5.9):

$$x - d_1,\ x,\ x + d_2. \tag{6.1}$$

A. If a complete curve is used in the simulation, the simulated balances will be level (Observation 5.9) so $d_1 = 0 = d_2$. Substituting in (6.1), the simulated balances are as follows:

$$x,\ x,\ x. \tag{6.2}$$

The value for x that minimizes the SSD is determined below using simple calculus:

$$\begin{aligned} SSD(x) &= (b_1-x)^2 + (b_2-x)^2 + (b_3-x)^2 \\ SSD'(x) &= -2(b_1-x) - 2(b_2-x) - 2(b_3-x) \\ &= -2b_1 + 2x - 2b_2 + 2x - 2b_3 + 2x \\ &= 6x - 2b_1 - 2b_2 - 2b_3. \end{aligned}$$

Setting SSD'(x) = 0 and solving for x:

$$6x - 2b_1 - 2b_2 - 2b_3 = 0$$

$$x = \frac{2b_1 + 2b_2 + 2b_3}{6} = \bar{b}.$$

Substituting in (6.2), the optimal simulated balances are as follows:

$$\bar{b},\ \bar{b},\ \bar{b}. \tag{6.3}$$

B. If a curve of unspecified length is used in the simulation, the simulated balances will increase (Observation 5.9) so $d_1 \geq 0$ and $d_2 \geq 0$ and the simulated balances as given in (6.1) are:

$$x - d_1,\ x,\ x + d_2.$$

The value for x that minimizes the SSD is determined below where d_1 and d_2 are any fixed constants:

$$\begin{aligned} SSD(x) &= (b_1-(x-d_1))^2 + (b_2-x)^2 + (b_3-(x+d_2))^2 \\ SSD'(x) &= -2(b_1-x+d_1) -2(b_2-x) -2(b_3-x-d_2) \\ &= -2b_1 + 2x -2d_1 -2b_2 + 2x -2b_3 + 2x + 2d_2 \\ &= 6x - 2b_1 - 2b_2 - 2b_3 - 2d_1 + 2d_2. \end{aligned}$$

Setting SSD'(x) = 0 and solving for x:

$$6x - 2b_1 - 2b_2 - 2b_3 - 2d_1 + 2d_2 = 0$$

$$x = \frac{2b_1+2b_2+2b_3}{6} + \frac{2d_1-2d_2}{6}$$

$$= \bar{b} + \frac{d_1-d_2}{3}.$$

Substituting in (6.1), the optimal simulated balances are as follows:

$$\left(\bar{b} + \frac{d_1-d_2}{3}\right) - d_1,\ \bar{b} + \frac{d_1-d_2}{3},\ \left(\bar{b} + \frac{d_1-d_2}{3}\right) + d_2. \qquad (6.4)$$

Note that if $d_1 = d_2 = 0$, the optimal simulated balances are found to be level at $\bar{b}$, which confirms part A above.

C. The SSD for the optimal balances (6.4) produced by a curve of unspecified length is shown below to be greater than or equal to the SSD for the optimal level balances (6.3) produced by a complete curve. It is concluded that level simulated balances at $\bar{b}$ are the best matching balances to level or decreasing actual balances.

The optimal simulated balances given by (6.4) can be considered to lie on a line with two segments (e.g., Figure 15). The slopes of the segments are represented by the annual increases $d_1 \geq 0$ and $d_2 \geq 0$ since the horizontal axis is partitioned into yearly intervals.

From Figure 15, it can be seen that the SSD for any such (broken) line of simulated balances can be decreased by rotating the line about the middle simulated balance until the line is horizontal. The SSD for this horizontal line can be further decreased by moving the line to $\bar{b}$. Since the SSD for level balances at $\bar{b}$ is less than the SSD for any optimal increasing simulated balances, the best matching simulated balances are level at $\bar{b}$.[5]

Observation 6.9: For 3-year bands after one life cycle of the earliest vintage in accounts with level installations and vintage lives decreasing by a fixed amount, the optimal SSD and CI can be calculated as follows:

$$SSD = 3\ VAR_b \qquad CI = \frac{\bar{b}}{SD_b}$$

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[5]The same conclusion can be reached by showing that the partial derivatives of the SSD with respect to d_1 and d_2 are positive and so the SSD is decreased by decreasing d_1 and d_2 to their lower limit of zero (see Appendix A).

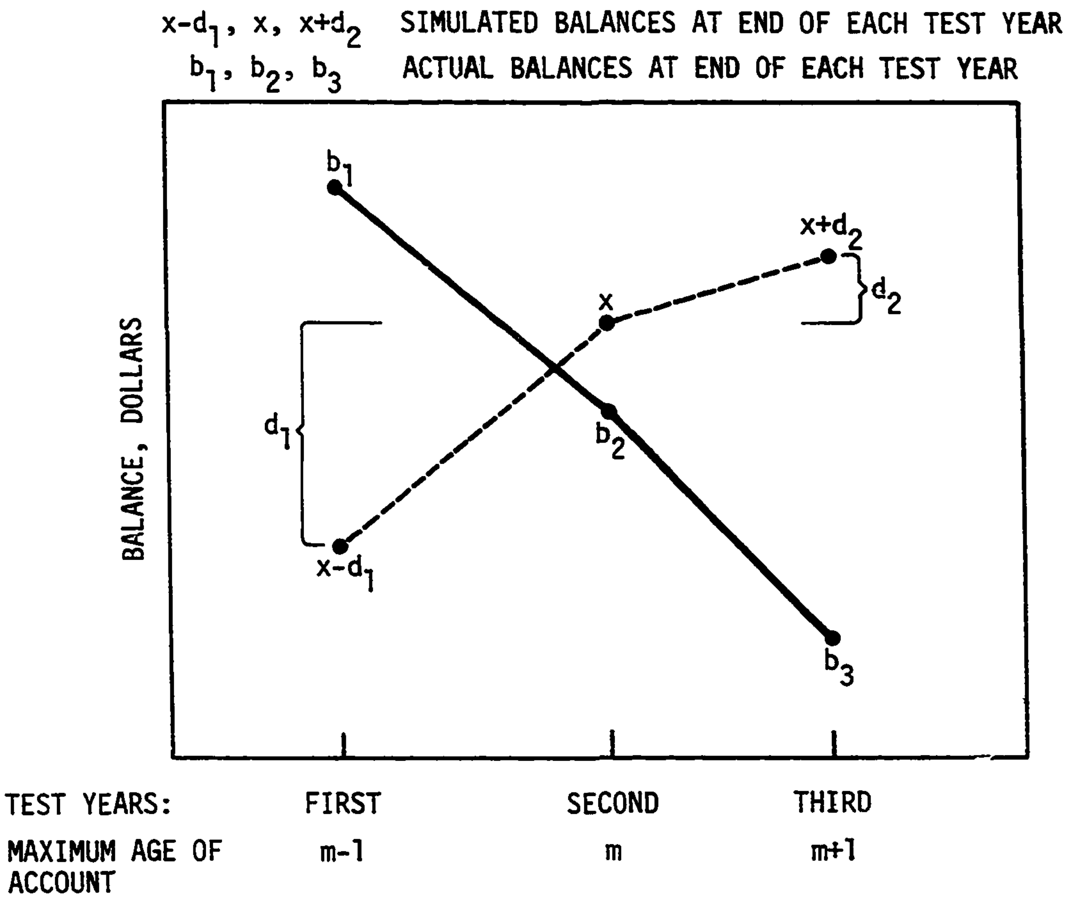


Figure 15. Actual and simulated balances at the end of each test year in a 3-year band

where VAR_b = variance of the band's actual balances

SD_b = standard deviation of the band's actual balances.

The variance and standard deviation of the actual balances in a 3-year band are given below:

$$VAR_b = \frac{\sum_{i=1}^{3} (b_i - \bar{b})^2}{3} \quad \text{so} \quad \sum_{i=1}^{3} (b_i - \bar{b})^2 = 3\ VAR_b. \tag{6.5}$$

$$SD_b = (VAR_b)^{1/2} \quad \text{so} \quad SD_b = \left[\frac{\sum_{i=1}^{3} (b_i - \bar{b})^2}{3}\right]^{1/2}. \tag{6.6}$$

In these accounts, retirements stabilized after one life cycle of the earliest vintage and remained so until after installations ceased. Since the retirements stabilized above the level installation amount, the actual balances decreased uniformly.

By Observation 6.8, the optimal simulated balances are level at $\bar{b}$. The SSD and CI for level simulated balances at $\bar{b}$ are given below:

$$SSD = \sum_{i=1}^{3} (b_i - \bar{b})^2 \tag{6.7}$$

$$\text{and} \quad CI = \frac{\bar{b}}{(\frac{SSD}{3})^{1/2}} = \frac{\bar{b}}{\left[\frac{\sum_{i=1}^{3} (b_i - \bar{b})^2}{3}\right]^{1/2}} \tag{6.8}$$

Substituting (6.5) in (6.7) and (6.6) in (6.8), the optimal SSD and CI are:

$$SSD = 3\ VAR_b \quad \text{and} \quad CI = \frac{\bar{b}}{SD_b}.$$

Therefore, the optimal SSD and CI can be determined from the variance (slope) and average of the actual balances. Because there is an upper limit on the CI in certain cases, it is questionable whether one index scale, such as Bauhan's, is applicable in all cases.

Observation 6.10: For 3-year bands in accounts with level installations and vintage lives decreasing by a fixed amount, the optimal SSD stabilized after which the optimal CI decreased for bands from the beginning to the end of the stability period.

As noted in the development of Observation 6.9, these accounts contained a period of stability in retirements during which the balances decreased uniformly and so had constant variance. As a result, the optimal SSD remained constant for all 3-year bands in this period of stability (Observation 6.9). Because the variance (and therefore the standard deviation) was stable during this period and $\bar{b}$ was decreasing, the optimal CI decreased for bands from the beginning to the end of the period (Observation 6.9).

Observation 6.11: For 3-year bands with decreasing balances in accounts with level installations, the curves that simulated optimal balances had an ASL = $\bar{b}$/INSTS and maximum life ratio (MLR) such that:

$$MLR \leq \frac{m}{\bar{b}/INSTS}.$$

For accounts with level installations, decreasing balances are matched best by simulated balances that are level at $\bar{b}$ (Observation 6.8).

To produce level balances, the curve must be complete by age m (Observation 5.9). A complete curve produces a balance $\overline{b}$ if it has ASL = $\overline{b}$/INSTS (Observation 6.3). For a curve with this ASL to be complete by age m, it must have a maximum life to average life ratio (MLR) such that

$$MLR \leq \frac{m}{\overline{b}/INSTS}.$$

For example, the following calculation can be made for the band in Table 16:

$$\frac{m}{\overline{b}/INSTS} = \frac{1953-1940+0.5}{9229/1000} = 1.46.$$

The highest ranked curves in Table 16 have ASL = $\overline{b}$/INSTS = 9229/1000 = 9.229. These curves also have MLR $\leq$ 1.46 as determined from Table 17. A curve with an MLR only slightly greater than 1.46 will also have a high rank if the curve's percent surviving at age m is negligible. Even the highest ranked curves in Table 16 have a poorly rated CI (by Bauhan's scale) because the level simulated balances, although optimum for this band, are not good matches to the decreasing balances.

When installations are level and balances are decreasing, the quotient calculated for a band indicates whether SPR will show a preference for high mode curves or will instead indicate multiple equally ranked curves, as discussed in the following observations.

Observation 6.12: For 3-year bands of decreasing balances in accounts with level installations, the high mode curves ranked highest when SPR indicated a ranking of curves.

Table 16. SPR results for 1952-54 band of R4-10,4 level installations account

Iowa curve type	ASL (years)	CI	REI
S_6	9.229	50	100.00
R_5	9.228	50	100.00
S_5	9.229	50	100.00
R_4	9.228	50	100.00
S_4	9.229	50	99.99
L_5	9.231	47	99.86
R_3	9.234	47	99.74
S_3	9.238	45	99.39
L_4	9.253	43	98.49
S_2	9.274	40	97.04
R_2	9.288	37	95.88
L_3	9.392	34	92.76
S_1	9.375	32	91.74
L_2	9.642	29	86.68
R_1	9.490	27	86.76
S_0	9.599	26	83.69
L_1	10.012	25	80.06
L_0	10.784	21	72.64
O_1	10.194	20	71.62
O_2	11.442	20	70.36
O_3	15.024	18	63.27
O_4	19.261	17	60.51

Table 17. Maximum life to average life ratios for the Iowa curves in increasing order

Iowa curve type	Maximum life ratio (MLR)
S_6	1.32
R_5	1.37
S_5	1.48
R_4	1.51
S_4	1.65
R_3	1.68
L_5	1.81
S_3	1.85
R_2	1.85
S_2	1.95
S_1	1.99
S_0	2.00
O_1	2.00
R_1	2.01
L_4	2.03
L_3	2.31
L_2	2.70
L_1	3.01
O_2	3.09
L_0	3.75
O_3	3.86
O_4	4.40

This follows from Observation 6.11 and Table 17 since the curves with the least MLR have the highest mode.

Observation 6.13: For 3-year bands of decreasing balances in accounts with level installations, the highest ranked curves in a family had the lowest ASLs when SPR indicated a ranking of curves.

This follows since in this case the highest ranked curves have high modes (Observation 6.12) and mode and ASL are inversely related for two SPR indicated curves from the same family (Observation 6.5).

Observation 6.14: For 3-year bands of decreasing balances in accounts with level installations, SPR will not give a unique solution if

$$\frac{m}{\bar{b}/\text{INSTS}} \geq 1.37.$$

If the above quotient calculated from the data equals 1.37, then all curves with MLR $\leq$ 1.37 will produce (identical) optimal simulated balances and therefore have equal, optimal rank (Observation 6.11). Since by Table 17 two curves have MLR $\leq$ 1.37, these two curves will have equal, optimal rank so SPR cannot give a unique solution for the band.

If the quotient calculated for a band exceeds 4.4, all curves will be rated equally as was true in Table 18 for which the following calculation can be made:

$$\frac{m}{\bar{b}/\text{INSTS}} = \frac{1968-1940+0.5}{5444/1000} = 5.24 > 4.4.$$

Table 18. SPR results for 1967-69 band of R4-10,4 level installations account

Iowa curve type	ASL (years)	CI	REI
O_1	5.445	27	100.00
O_4	5.461	27	100.00
O_3	5.528	27	100.00
S_2	5.444	27	100.00
R_4	5.445	27	100.00
S_4	5.444	27	100.00
S_5	5.444	27	100.00
R_3	5.445	27	100.00
L_5	5.446	27	100.00
R_5	5.445	27	100.00
L_3	5.445	27	100.00
L_4	5.444	27	100.00
L_1	5.445	27	100.00
L_0	5.446	27	100.00
R_1	5.446	27	100.00
O_2	5.473	27	100.00
S_0	5.445	27	100.00
S_1	5.444	27	100.00
R_2	5.445	27	100.00
L_2	5.444	27	100.00
S_3	5.444	27	100.00
S_6	5.444	27	100.00

<u>Observation 6.15</u>: For 3-year bands of decreasing balances in accounts in which installations were level or increasing by a fixed amount (CHGE), the optimal simulated balances for curves that were complete at age m-1 were:

$$\bar{b} - \text{CHGE} \times \text{ASL},\ \bar{b},\ \bar{b} + \text{CHGE} \times \text{ASL}$$

where CHGE $\geq$ 0.

The following formulas can be developed for the annual increase in simulated balances (see Appendix B):

$$d_1 = \text{INSTS}_0 \frac{PS_m}{100} + \text{CHGE} \sum_{i=1}^{m-1} \frac{PS_i}{100} \qquad (6.9)$$

$$d_2 = \text{INSTS}_0 \frac{PS_{m+1}}{100} + \text{CHGE} \sum_{i=1}^{m} \frac{PS_i}{100} \qquad (6.10)$$

where INSTS_0 = installation amount for earliest vintage

CHGE = annual fixed amount change in installations

(CHGE $\geq$ 0 for this observation).

If the curve is complete at age m-1, then the following hold (see Iowa Type Survivor Curves, Chapter I):

$$PS_m = PS_{m+1} = 0. \qquad (6.11)$$

$$\sum_{i=1}^{m-1} \frac{PS_i}{100} = \text{ASL}. \qquad (6.12)$$

Substituting (6.11) and (6.12) in (6.9) and (6.10):

$$d_1 = 0 + CHGE \times ASL = CHGE \times ASL. \tag{6.13}$$

$$d_2 = 0 + CHGE \left(ASL + \frac{PS_m}{100}\right) = CHGE \times ASL. \tag{6.14}$$

In part B of Observation 6.8, the optimal simulated balances were found to be given by (6.4), regardless of the pattern of installations. Substituting (6.13) and (6.14) in (6.4) gives the optimal pattern for complete curves:

$$\bar{b} - CHGE \times ASL,\ \bar{b},\ \bar{b} + CHGE \times ASL. \tag{6.15}$$

If CHGE = 0 (level installations), then the optimal balances are level at $\bar{b}$ as was found in part A of Observation 6.8. If CHGE > 0 (increasing installations), then the simulated balances lie on a straight line with positive slope. The SSD for this line of balances can be decreased by decreasing the slope of the line. Since CHGE is fixed in an account, the slope of the line is least for curves with least ASL. This results in a ranking of the complete curves by increasing ASL (Table 19). By Observation 6.5, this is equivalent to a ranking by curve mode from highest to lowest mode.

Observation 6.16: Simulated balances for 3-year bands can decrease if installations decrease.

By Equation 6.9, which was developed with no restriction on the pattern of installations, the change in simulated balances between the first and second test years is given below, where CHGE < 0 since installations are decreasing:

Table 19. SPR results for 1967-69 band of decreasing balances of R4-10,5.5 incrementally increasing installations account

Iowa curve type	ASL (years)	CI	REI
S_6	6.443	58	100.00
S_5	6.444	55	100.00
R_5	6.446	55	100.00
L_5	6.447	55	100.00
S_4	6.448	55	100.00
R_4	6.451	55	100.00
L_4	6.454	55	100.00
S_3	6.455	55	100.00
R_3	6.459	55	100.00
S_2	6.464	55	100.00
L_3	6.467	55	100.00
R_2	6.474	55	100.00
S_1	6.477	55	100.00
L_2	6.484	55	100.00
R_1	6.494	55	100.00
S_0	6.494	55	100.00
L_1	6.506	55	100.00
O_1	6.522	55	100.00
L_0	6.538	55	100.00
O_2	6.593	55	100.00
O_3	6.751	55	100.00
O_4	6.769	55	100.00

$$d_1 = \text{INSTS}_0 \frac{PS_m}{100} + \text{CHGE} \sum_{i=1}^{m-1} \frac{PS_i}{100}.$$

If $\text{INSTS}_0 \frac{PS_m}{100} < |\text{CHGE}| \sum_{i=1}^{m-1} \frac{PS_i}{100}$, simulated balances will decrease.

If $\text{INSTS}_0 \frac{PS_m}{100} = |\text{CHGE}| \sum_{i=1}^{m-1} \frac{PS_i}{100}$, simulated balances will be level.

If $\text{INSTS}_0 \frac{PS_m}{100} > |\text{CHGE}| \sum_{i=1}^{m-1} \frac{PS_i}{100}$, simulated balances will increase.

Observation 6.17: For 3-year bands in accounts with decreasing installations, the maximum decrease in simulated balances was achieved by complete curves.

By Equation 6.9, the change in simulated balances between the first and second test years is given below, where CHGE < 0 since installations are decreasing:

$$d_1 = \text{INSTS}_0 \frac{PS_m}{100} + \text{CHGE} \sum_{i=1}^{m-1} \frac{PS_i}{100}. \tag{6.16}$$

If a survivor curve is complete at age m-1, the following hold (see Iowa Type Survivor Curve, Chapter I):

$$PS_m = 0 \tag{6.17}$$

and $$\sum_{i=1}^{m-1} \frac{PS_i}{100} = \text{ASL}. \tag{6.18}$$

Substituting (6.17) and (6.18) in the formula for d_1 (6.16):

$$d_1 = 0 + CHGE \times ASL. \qquad (6.19)$$

If, on the other hand, the survivor curve is not complete, i.e., $m-1 <$ maximum life of the curve, then

$$PS_m \neq 0 \qquad (6.20)$$

$$\text{and} \quad \sum_{i=1}^{m-1} \frac{PS_i}{100} < ASL. \qquad (6.21)$$

The effect of (6.20) and (6.21) on (6.16) is to increase the first (positive) term and decrease the second (negative) term so d_1 becomes less negative. Therefore, d_1 is most negative and has the value given by (6.19) when the curve used in the simulation is complete by age m-1. Likewise, d_2 is most negative and has the value given by (6.19) when the curve is complete.

When actual balances are steeply decreasing, this may result in a preference for high mode curves since they may be the only curves able to enclose the required area and be complete by the maximum age of the account.

Observation 6.18: For 3-year bands of decreasing balances in accounts with decreasing installations, the ASL and curve mode were directly related for two complete SPR indicated curves from the same family.

Consider a high mode and a low mode curve from the same family and with the same ASL as in Figure 14. The discussion below shows that if the complete curves are used to produce the same balance, the ASLs have the following relationship:

ASL of high mode curve $\geq$ ASL of low mode curve.

The simulated balances produced by complete curves lie on a straight line with annual, negative change d (i.e., $d_1 = d_2 = d$ and $d < 0$) (Equation 6.19). By Equation 6.4, the development of which placed no conditions on the sign of d, the optimal simulated balances are as follows:

$$\bar{b} - d, \bar{b}, \bar{b} + d.$$

Since the two curves in Figure 14 have the same ASL, they enclose equal area (see Iowa Type Survivor Curve, Chapter I). They will not produce the same balance $\bar{b}$ for the middle test year because the tail of a survivor curve is weighted more heavily than the front part when installations are decreasing and the low mode curve is above the high mode curve after the two curves intersect. To offset the effect of the weighting, the tail of the low mode curve can be lowered by decreasing the curve's ASL. The ASL of the low mode curve will then be less than that of the high mode curve.

Summary of Phase II Results

Listing of observations

6.1. For the accounts in which all vintages had the same life characteristics, SPR detected the life characteristics, except during stability.

6.2. For accounts in which all vintages had equal installations and the same life characteristics, SPR gave indeterminant results when the property was at stability.

6.3. For accounts with level installations, a complete survivor curve that simulated a balance equal to the actual annual balance had average service life (ASL) equal to the quotient of the actual balance and the level installation amount.

6.4. For accounts with a trend in vintage lives, SPR detected the trend.

6.5. For accounts with level or increasing installations, the ASL and the mode were inversely related fro two SPR indicated curves from the same family.

6.6. For 3-year bands in accounts with level installations (INSTS), the following held for the survivor curves that produced simulated balances matching the actual balances:

a. The area enclosed by the survivor curve stubbed at age m multiplied by INSTS equaled the actual balance of the second (middle) test year.

b. The height (percent surviving) of the survivor curve at ages m and m+1 multiplied by INSTS equaled the increase in actual balances between the first and second and between the second and third test years, respectively.

6.7. For 3-year bands in accounts with level installations, steeply (gradually) increasing actual balances (b_1, b_2, b_3) were matched best by a survivor curve that enclosed area equal to b_1/INSTS by age m-1 and had a high (low) height at ages m and m+1.

6.8. Three-year bands of level or decreasing balances in accounts with level installations were matched best by level simulated balances equal to $\bar{b}$, the average of the band's actual balances.

6.9. For 3-year bands after one life cycle of the earliest vintage in accounts with level installations and vintage lives decreasing by a fixed amount, the optimal SSD and CI can be calculated as follows:

$$SSD = 3\ VAR_b \qquad CI = \frac{\bar{b}}{SD_b}$$

where VAR_b = variance of the band's actual balances

SD_b = standard deviation of the band's actual balances.

6.10. For 3-year bands in accounts with level installations and vintage lives decreasing by a fixed amount, the optimal SSD stabilized after which the optimal CI decreased for bands from the beginning to the end of the stability period.

6.11. For 3-year bands with decreasing balances in accounts with level installations, the curves that simulated optimal balances had an ASL = $\bar{b}$/INSTS and maximum life ratio (MLR) such that:

$$MLR \leq \frac{m}{\bar{b}/INSTS}.$$

6.12. For 3-year bands of decreasing balances in accounts with level installations, the high mode curves ranked highest when SPR indicated a ranking of curves.

6.13. For 3-year bands of decreasing balances in accounts with level installations, the highest ranked curves in a family had the lowest ASL's when SPR indicated a ranking of curves.

6.14. For 3-year bands of decreasing balances in accounts with level installations, SPR will not give a unique solution if

$$\frac{m}{\bar{b}/\mathrm{INSTS}} \geq 1.37.$$

6.15. For 3-year bands of decreasing balances in accounts in which installations were level or increasing by a fixed amount (CHGE), the optimal simulated balances for curves that were complete at age m-1 were:

$$\bar{b} - \mathrm{CHGE} \times \mathrm{ASL},\ \bar{b},\ \bar{b} + \mathrm{CHGE} \times \mathrm{ASL}$$

where CHGE $\geq$ 0.

6.16. Simulated balances for 3-year bands can decrease if installations decrease.

6.17. For 3-year bands in accounts with decreasing installations, the maximum decrease in simulated balances was achieved by complete curves.

6.18. For 3-year bands of decreasing balances in accounts with decreasing installations, the ASL and curve mode were directly related for two complete SPR indicated curves from the same family.

Discussion of observations

SPR is a fixed life model. It assumes that all property has the same life characteristics which can be represented by an Iowa survivor curve. If these assumptions are met, the model will reflect the property's life characteristics during periods of instability. For accounts in which the life characteristics vary by vintage, because SPR cannot likewise vary the survivor curve by vintage, it attempts to match conditions created by variable life property by adjusting the curve type or ASL of a single survivor curve. Use of this fixed life model with variable life data can cause the model to make extreme adjustments. For example, since the model is unable to produce decreasing balances unless installations are decreasing, it adjusts by using high mode curves with low ASLs. The model adjusts to match low, gradually increasing balances in accounts with level installations and variable lives by using origin moded curves with high ASLs.

Not only can the same extreme curves be indicated for accounts that differ from each other in life characteristics, but they can also be indicated for different bands in the same account. If applied to variable life data, the SPR model may fail to indicate a unique curve, such as when the account is young or when balances are decreasing and installations are not decreasing.

Based on this study, SPR cannot be relied upon to indicate life characteristics for variable life property. Although the model can detect a trend in vintage lives, it has a bias to high mode curves with low ASL or to origin moded curves with high ASL under certain definable conditions in heterogeneous property. Under other conditions, the model indicates several equal and highly ranked curves. The model is intended for and gives consistent and appropriate results for property which has the same life characteristics.

CHAPTER VII. DISCUSSION OF PHASE III RESULTS

In Phase III, the impact of the difference in survivor curves indicated by the SPR and RR models in Phase I was investigated by comparing the depreciation (see Depreciation Models, Chapter I) calculated using the different curves. This phase examined two of the accounts with decreasing vintage lives that were examined in Phase I along with two accounts with increasing vintage lives. All accounts had level installations.

As explained in Chapter IV, the survivor curves indicated by the life analysis models were used to calculate depreciation for 5 years beginning with the year the life analysis was made (the most recent year in the experience band). For the calculation of accumulated depreciation, the RR model's survivor curve was applied to the actual vintage balances whereas each SPR indicated curve was applied to the actual vintage balances (SPR-A) and to simulated vintage balances (SPR-S). Since only the actual balance is used to calculate the annual accrual, only one accrual calculation (arbitrarily labeled SPR-S) was made using the SPR indicated curve. The annual and accumulated depreciation associated with each life analysis model were compared to the actual (book) depreciation which was calculated by the vintage group procedure using the vintage curves that were used to generate the original data.

The accumulated depreciation comparisons are discussed first, followed by a comparison of annual accruals. The comparison of annual depreciation associated with the different curves for a given year is equivalent to a comparison of the curves' average service lives (ASLs) since the ASL was the only parameter that changed as the accrual associated with different curves was calculated using the straight line method and average life procedure:

$$\text{annual accrual} = \frac{\text{average actual balance}}{\text{ASL}}.$$

Observations

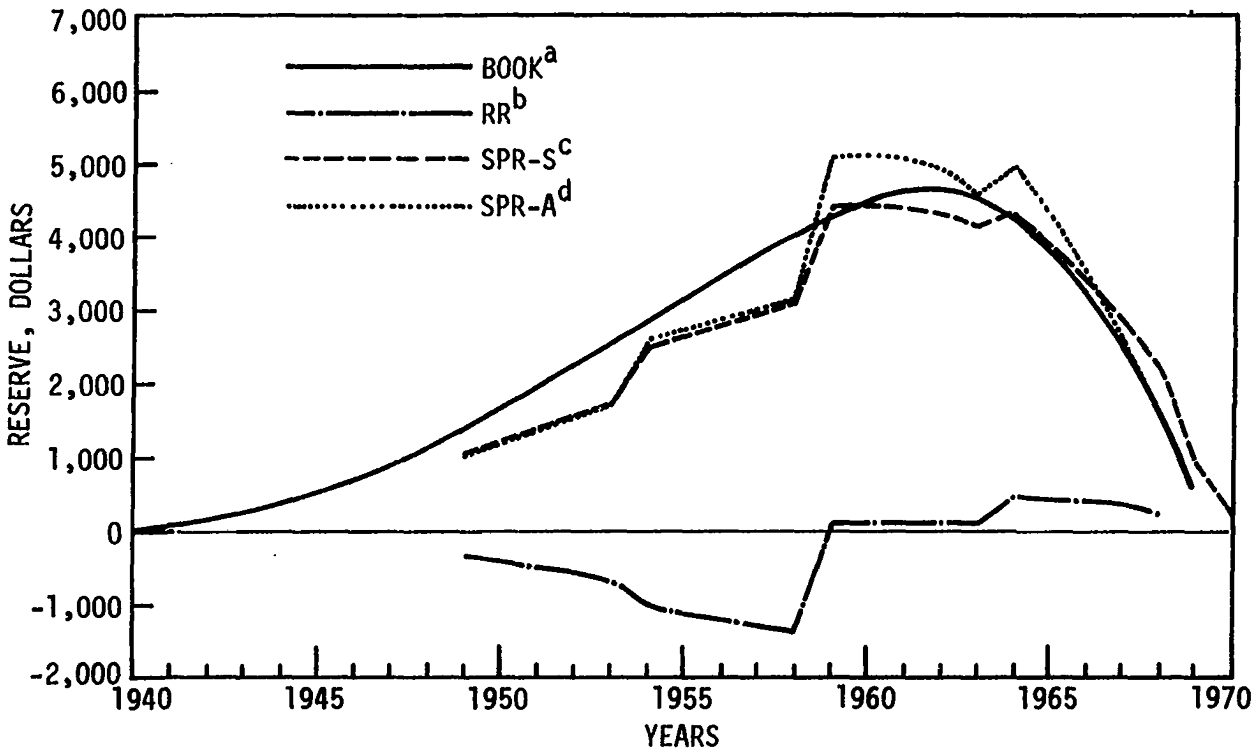
Observation 7.1: For the accounts with decreasing vintage lives and level installations, the accumulated depreciation based on the survivor curves indicated by the SPR model exceeded that based on RR model curves and was closer to the book depreciation.

This follows from analyzing the depreciation for the 01-15,5 and S4-15,5 accounts (Figures 16 and 17) which was based on the survivor curves listed in Table 20. Since SPR did not indicate a unique curve for bands after 1958 for both accounts, a curve was arbitrarily chosen from those with the highest rank; therefore, the depreciation graph could be changed by choosing another curve.

This observation implies that the RR model may not indicate the "correct" curve to use for depreciation purposes.

Observation 7.2: For the accounts with increasing vintage lives and level installations, the accumulated depreciation based on survivor curves indicated by the RR model exceeded that based on the SPR model curves during the installation period and was closer to the book depreciation for almost all of this period.

This follows from analyzing the depreciation for the 01-5,15 and S4-5,15 accounts (Figures 18 and 19) which are based on survivor curves



[a]Calculated using the vintage survivor curves that were used to generate the data set

[b]Calculated using survivor curves indicated by RR model with actual age distribution

[c]Calculated using survivor curves indicated by SPR model with simulated age distribution

[d]Calculated using survivor curves indicated by SPR model with actual age distribution

Figure 16. Accumulated depreciation per book and per survivor curves indicated by RR and SPR analyses of 01-15,5 level installations account

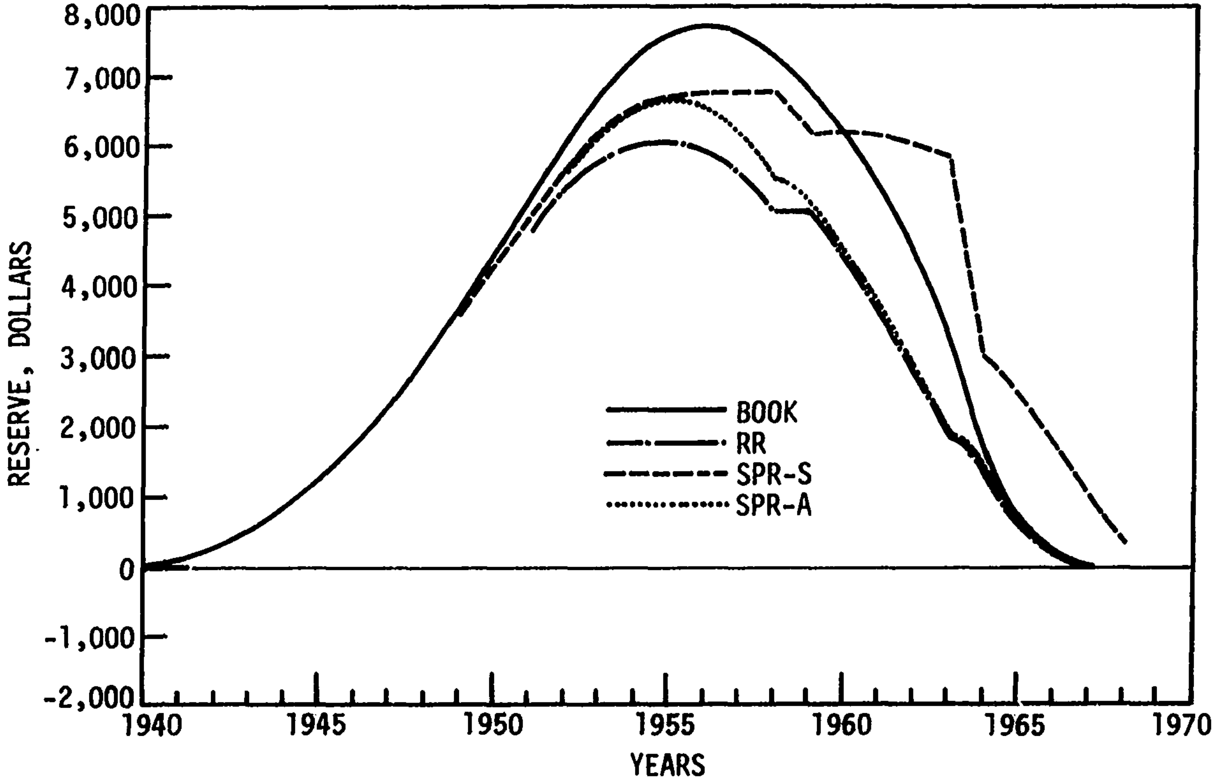


Figure 17. Accumulated depreciation per book and per survivor curves indicated by RR and SPR analyses of S4-15,5 level installations account

Table 20. Highest ranked curves by RR and SPR models for "15,5" level installations accounts by band in which the curves were used to calculate depreciation

Band in which curves used	01-15,5		S4-15,5	
	RR	SPR	RR	SPR
1949-53	O_4-27.9	O_2-15.4	S_4-14.6	R_5-14.0
1954-58	O_4-22.3	L_0-13.1	L_4-14.6	R_5-13.7
1959-63	O_3-12.8	R_1-11.2	S_2-11.2	R_5-12.5
1964-68	O_3-8.2	L_4-10.7	L_3-6.9	S_5-8.0
1969-70	--[a]	S_6-9.9	--[a]	S_5-5.5

[a]Survivor curves were not calculated since data were not available to calculate retirement rates for the curves' early age intervals.

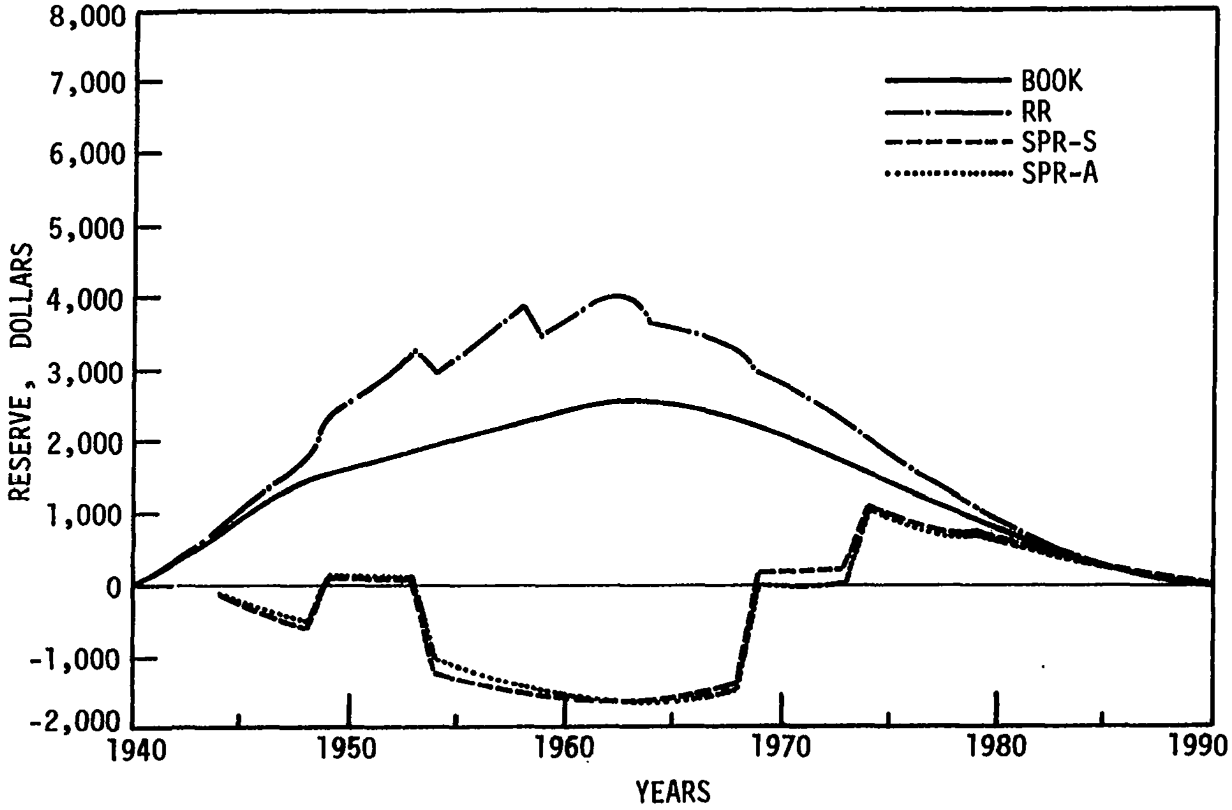


Figure 18. Accumulated depreciation per book and per survivor curves indicated by RR and SPR analyses of 01-5,15 level installations account

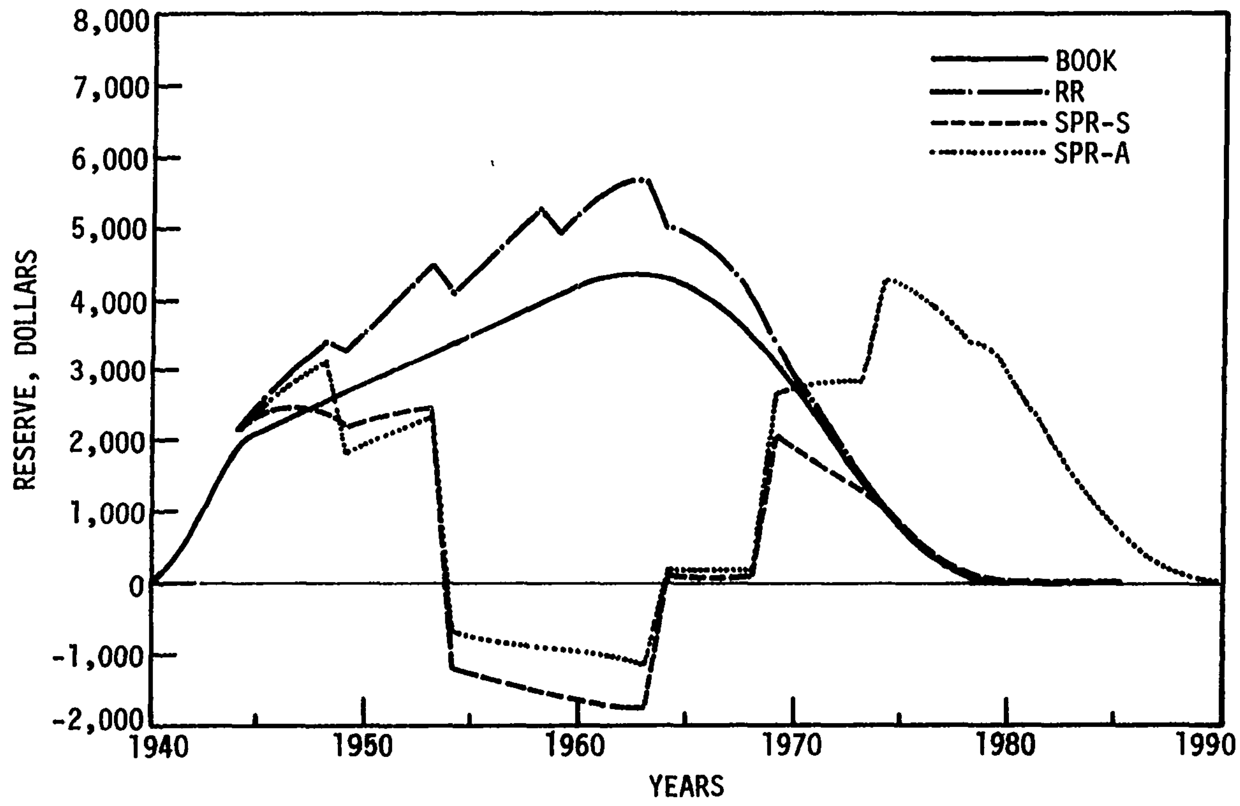


Figure 19. Accumulated depreciation per book and per survivor curves indicated by RR and SPR analyses of S4-5,15 level installations account

listed in Table 21. This observation implies that the SPR model may not indicate the "correct" curve to use for depreciation purposes.

Observation 7.3: For the account with decreasing vintage lives and level installations, the annual accrual based on survivor curves indicated by both the RR and SPR models was below the book accrual. In other words, the ASL indicated by both life analysis models was greater than the book (reciprocally weighted) ASL. During the installation period, the depreciation associated with the SPR model was closer to the book depreciation.

This follows from analyzing Figure 20.

Observation 7.4: For the account with increasing vintage lives and level installations, the annual accrual associated with the SPR model was closer to the book depreciation than was the RR associated depreciation except at the beginning and end of the account.

This follows from analyzing Figure 21.

Summary of Phase III Results

Listing of observations

7.1. For the accounts with decreasing vintage lives and level installations, the accumulated depreciation based on the survivor curves indicated by the SPR model exceeded that based on RR model curves and was closer to the book depreciation.

7.2. For the accounts with increasing vintage lives and level installations, the accumulated depreciation based on survivor curves indicated by the RR model exceeded that based on the SPR model curves

Table 21. Highest ranked curves by RR and SPR models for "5,15" level installations accounts by band in which the curves were used to calculate depreciation

Band in which curves used	O1-5,15		S4-5,15	
	RR	SPR	RR	SPR
1944-48	O_1-5.5	O_4-11.5	L_5-5.0	S_4-5.0
1949-53	R_1-6.0	O_3-8.8	S_4-6.0	L_1-6.9
1954-58	R_1-7.5	O_4-11.0	L_5-7.7	O_4-12.9
1959-63	R_1-9.2	O_4-11.9	R_5-9.4	O_4-13.7
1964-68	R_1-10.8	O_4-12.6	R_5-11.1	O_3-12.5
1969-73	R_1-11.5	O_3-11.5	S_4-12.0	L_0-11.4
1974-78	R_1-12.1	O_2-11.8	S_4-12.6	R_2-12.5
1979-83	S_0-12.5	L_0-12.5	L_4-12.6	S_3-13.6
1984-88	S_0-12.6	L_1-13.8	L_4-12.6	L_5-14.2
1989-90	S_0-12.6	S_1-16.7	L_4-12.6	L_5-14.2

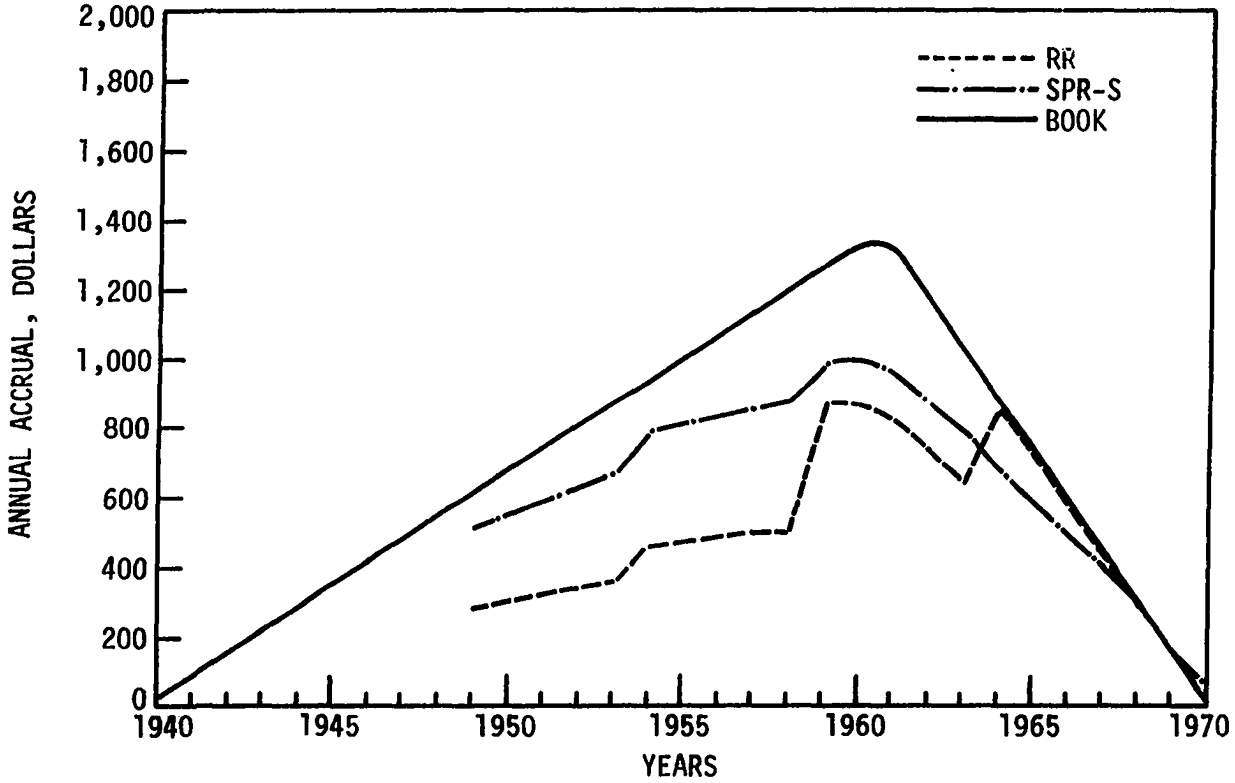


Figure 20. Annual accrual per book and per survivor curves indicated by RR and SPR analyses of 01-15,5 level installations account

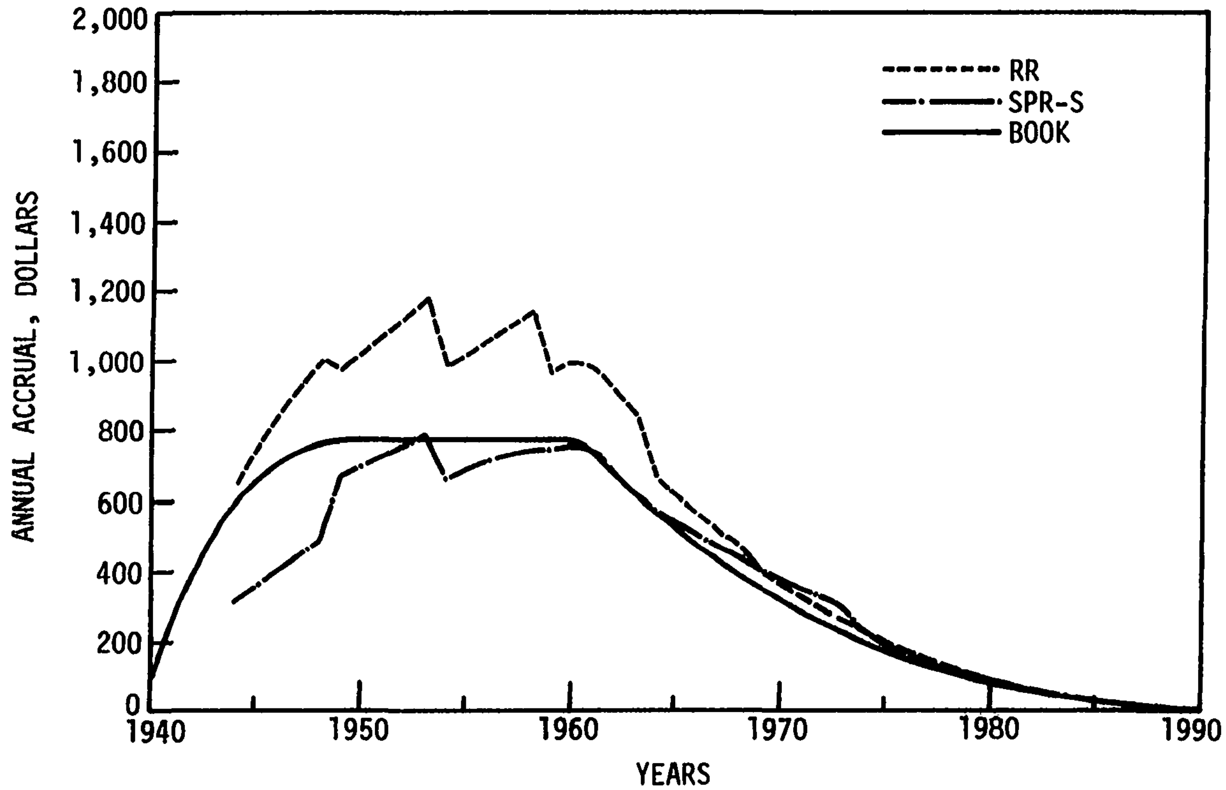


Figure 21. Annual accrual per book and per survivor curves indicated by RR and SPR analyses of 01-5,15 level installations account

during the installation period and was closer to the book depreciation for almost all of this period.

7.3. For the account with decreasing vintage lives and level installations, the annual accrual based on survivor curves indicated by both the RR and SPR models was below the book accrual. In other words, the ASL indicated by both life analysis models was greater than the book (reciprocally weighted) ASL. During the installation period, the depreciation associated with the SPR model was closer to the book depreciation.

7.4. For the account with increasing vintage lives and level installations, the annual accrual associated with the SPR model was closer to the book depreciation than was the RR associated depreciation except at the beginning and end of the account.

Discussion of observations

For accounts with level installations, neither the RR nor the SPR model consistently indicated survivor curves that generated accumulated depreciation matching the actual (book) depreciation. The ASLs the curves indicated by the SPR model were closer almost everywhere to the book (reciprocally weighted) ASL than were the ASLs indicated by the RR model.

If the survivor curves are being estimated for use in calculating depreciation and if the actual (book) depreciation calculated by the vintage group procedure using the property's known life characteristics is considered to be the standard, then based on this study, neither model can be relied upon to consistently meet the standard. If the book reciprocally weighted ASL is considered to be the standard, SPR is a more reliable indicator of ASL than of dispersion.

CHAPTER VIII. CONCLUSIONS AND RECOMMENDATIONS

This study tested the simulated plant-record (SPR) balances model against two standards: the survivor curves indicated by the retirement rate (RR) model and the depreciation calculated using the property's known life characteristics. The performance of the balances model with respect to these two standards is summarized below. Conclusions are also included with respect to the appropriate application of the model in view of its idiosyncrasies. The conclusions were developed from and so are conditioned upon the characteristics of the accounts used in this study. Following the conclusions are recommendations for future investigation of SPR.

1. If the survivor curve indicated by the RR model is considered to be the standard, then the SPR model cannot be relied upon to meet the standard unless all property has the same life characteristics.

2. If depreciation calculated by the straight-line method and vintage group procedure using the property's known life characteristics is considered to be the standard, then neither the SPR nor the RR model, used with the broad group procedure, can be relied upon to consistently meet the standard.

3. If the reciprocally weighted ASL resulting from the depreciation calculated by the straight-line method and vintage group procedure using the property's known life characteristics is considered to be the standard, then the SPR model is a more reliable indicator of ASL than of retirement dispersion pattern.

4. SPR reflects the property's life characteristics when all property has the same life characteristics.

5. SPR cannot be relied upon to indicate life characteristics when all property does not have the same life characteristics (i.e., heterogeneous property).

6. SPR indicates a preference for curves with high mode and low ASL or with mode at the origin and high ASL under certain definable conditions in heterogeneous property.

7. SPR does not indicate a unique curve under certain definable conditions.

8. For property with a trend in vintage lives, SPR detected the trend.

The SPR balances model indicates an appropriate curve when the life characteristics of the property are homogeneous, but it has serious problems when applied to property with heterogeneous life characteristics. Based on this study, certain conditions under which SPR will give inappropriate results can be identified and avoided by choosing test bands in consideration of the pattern of installations and balances and the maximum age of the account. This is not to say that avoiding these conditions will assure that appropriate curves will be indicated by SPR.

This study was intended to develop conclusions where possible but also to uncover areas for future work, such as those listed below:

1. A comparison could be made of SPR results produced using:
   a. Bands of different widths.
   b. Bands in different periods.

c. Accounts with patterns of installations, vintage life or curve type different from those used in this study.

d. Nonuniform data that more approximates real world conditions.

2. The relationship between the property's life characteristics and the pattern of installations and balances could be studied and compared to the SPR results.

3. The appropriateness of the SSD criterion and the usefulness of the conformance index could be evaluated.

4. Consideration could be given to the development of indices (like the retirement experience index) that would reflect the age of the oldest and youngest significant vintages at the beginning and end of the test band.

The problems associated with finding an average curve for property with heterogeneous life characteristics can be avoided if depreciation is calculated by vintage. A semiactuarial model, like computed mortality (CM), is needed that permits the development of vintage survivor curves. Because of the lack of widely available information about CM or published empirical tests of it, it is recommended that the CM model be investigated. A desirable product of this investigation would be the location or development of an adaptable and transportable computer program for CM like the programs that are currently available for SPR.

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ACKNOWLEDGMENTS

The completion of this research was due to the encouragement, cooperation, and assistance provided by many individuals for which the author is deeply grateful. The author would like to thank Dr. Harold A. Cowles for his direction and cooperation throughout her studies at Iowa State University (ISU). The author is grateful to Dr. W. Chester Fitch, President, Depreciation Programs, Inc., for his direction and critical comments and for the use of his computer research account at Western Michigan University (WMU). Dr. Frank K. Wolf is thanked for making available the graphics facilities of the Department of Industrial Engineering, WMU. The author appreciates the generosity of Robert E. Boughner, Acting Dean, College of Engineering and Applied Sciences, WMU, in providing her with an office.

For financial support, recognition is due the Power Affiliate Program, ISU; Depreciation Programs, Inc.; and the Center for Depreciation Studies, WMU.

For encouragement and critical comments, the author is grateful to J. B. Keeler. The author thanks her mother for believing in and encouraging her.

APPENDIX A.

MATHEMATICAL SUPPORT FOR OBSERVATION 6.8

In order to conclude that the optimal simulated balances lie on a horizontal line, it is necessary to show that the SSD for the optimal horizontal line is less than the SSD for any other line of simulated balances. This is done by expressing the SSD as a function of the slopes d_1 and d_2 of the two segments of a broken line of simulated balances (Figure 15) and showing that the SSD is directly related to each slope. In other words, decreasing either slope results in a decrease in the SSD. The optimal SSD then results when each slope is equal to its minimum, which is zero according to the constraints of the SPR model. The following notation from the Notation page is used along with expression (6.4):

b_i actual balance for ith test year

d_i change in simulated balances between ith and (i+1)st test year

optimal simulated balances for a 3-year band:

$$\bar{b} + \frac{d_1-d_2}{3} - d_1,\ \bar{b} + \frac{d_1-d_2}{3},\ \bar{b} + \frac{d_1-d_2}{3} + d_2.$$

$$SSD(d_1,d_2) = \left[b_1-(\bar{b}+\frac{d_1-d_2}{3} - d_1)\right]^2 + \left[b_2-(\bar{b}+\frac{d_1-d_2}{3})\right]^2 + \left[b_3-(\bar{b}+\frac{d_1-d_2}{3}+d_2)\right]^2$$

$$= \left[b_1-(\bar{b}+\frac{d_1-d_2-3d_1}{3})\right]^2+\left[b_2-(\bar{b}+\frac{d_1-d_2}{3})\right]^2+\left[b_3-(\bar{b}+\frac{d_1-d_2+3d_2}{3})\right]^2$$

$$= \left[b_1-\bar{b}-\frac{-2d_1-d_2}{3}\right]^2+\left[b_2-\bar{b}-\frac{d_1-d_2}{3}\right]^2+\left[b_3-\bar{b}-\frac{d_1+2d_2}{3}\right]^2$$

$$= \left[b_1-\bar{b}+\frac{2}{3}d_1+\frac{1}{3}d_2\right]^2+\left[b_2-\bar{b}-\frac{1}{3}d_1+\frac{1}{3}d_2\right]^2+\left[b_3-\bar{b}-\frac{1}{3}d_1-\frac{2}{3}d_2\right]^2.$$

$$\frac{\partial SSD}{\partial d_1} = 2\left[\frac{2}{3}(b_1-\bar{b}+\frac{2}{3}d_1+\frac{1}{3}d_2)-\frac{1}{3}(b_2-\bar{b}-\frac{1}{3}d_1+\frac{1}{3}d_2)-\frac{1}{3}(b_3-\bar{b}-\frac{1}{3}d_1-\frac{2}{3}d_2)\right]$$

$$= \frac{2}{3}\left[2b_1-2\bar{b}+\frac{4}{3}d_1+\frac{2}{3}d_2-b_2+\bar{b}+\frac{1}{3}d_1-\frac{1}{3}d_2-b_3+\bar{b}+\frac{1}{3}d_1+\frac{2}{3}d_2\right]$$

$$= \frac{2}{3}\left[2b_1-b_2-b_3\right]+\frac{2}{3}\left[2d_1+d_2\right].$$

Now

$2b_1-b_2-b_3 = (b_1-b_2)+(b_1-b_3) > 0$ since $b_1 > b_2 > b_3$ because actual balances are decreasing

and

$2d_1+d_2 > 0$ since $d_1 > 0$ and $d_2 > 0$

so

$$\frac{\partial SSD}{\partial d_1} > 0.$$

$$\frac{\partial SSD}{\partial d_2} = 2\left[\frac{1}{3}(b_1-\bar{b}+\frac{2}{3}d_1+\frac{1}{3}d_2)+\frac{1}{3}(b_2-\bar{b}-\frac{1}{3}d_1+\frac{1}{3}d_2)-\frac{2}{3}(b_3-\bar{b}-\frac{1}{3}d_1-\frac{2}{3}d_2)\right]$$

$$= \frac{2}{3}\left[b_1-\bar{b}+\frac{2}{3}d_1+\frac{1}{3}d_2+b_2-\bar{b}-\frac{1}{3}d_1+\frac{1}{3}d_2-2b_3+2\bar{b}+\frac{2}{3}d_1+\frac{4}{3}d_2\right]$$

$$= \frac{2}{3}\left[b_1+b_2-2b_3\right]+\frac{2}{3}\left[d_1+2d_2\right].$$

Now

$b_1+b_2-2b_3=(b_1-b_3)+(b_2-b_3) > 0$ since $b_1 > b_2 > b_3$ because actual balances are decreasing

and

$d_1+2d_2 > 0$ since $d_1 > 0$ and $d_2 > 0$

so

$$\frac{\partial SSD}{\partial d_2} > 0.$$

APPENDIX B.

MATHEMATICAL SUPPORT FOR OBSERVATION 6.15

In order to determine the optimal simulated balances produced by complete curves for a 3-year test band, it is necessary to determine the annual change in simulated balances. The annual change is calculated by differencing formulas for the simulated balances for each test year. The following notation from the Notation page is used:

d_i	change in simulated balances between ith and (i+1)st test year
x	simulated balance for the second (middle) test year
m-1	maximum age of the account at the end of the first test year
m	maximum age of the account at the end of the second (middle) test year
m+1	maximum age of the account at the end of the third test year
PS_i	percent surviving at age i
$INSTS_0$	installation amount for the earliest vintage
CHGE	annual (constant) change in installations

$$x-d_1 = \sum_{i=1}^{m-1} (INSTS_0 + (m-1-i)CHGE) \frac{PS_i}{100} = \sum_{i=1}^{m-1} \frac{PS_i}{100} INSTS_0 + (m)CHGE \sum_{i=1}^{m-1} \frac{PS_i}{100}$$

$$- CHGE \sum_{i=1}^{m-1} \frac{PS_i}{100} - CHGE \sum_{i=1}^{m-1} \frac{PS_i}{100} i$$

$$= \sum_{i=1}^{m} \frac{PS_i}{100} INSTS_0 - INSTS_0 \frac{PS_m}{100} + (m)CHGE \sum_{i=1}^{m} \frac{PS_i}{100} - (m)CHGE \frac{PS_m}{100}$$

$$- CHGE \sum_{i=1}^{m} \frac{PS_i}{100} + CHGE \frac{PS_m}{100} - CHGE \sum_{i=1}^{m} \frac{PS_i}{100} i + (m)CHGE \frac{PS_m}{100}$$

$$x = \sum_{i=1}^{m} \frac{PS_i}{100} (INSTS_0 + (m-i)CHGE) = \sum_{i=1}^{m} \frac{PS_i}{100} INSTS_0 + (m)CHGE \sum_{i=1}^{m} \frac{PS_i}{100}$$

$$- CHGE \sum_{i=1}^{m} \frac{PS_i}{100} i = x_{\text{level installations}} + CHGE \sum_{i=1}^{m} \frac{PS_i}{100} (m-i)$$

$$x+d_2 = \sum_{i=1}^{m+1} \frac{PS_i}{100} (INSTS_0 + (m+1-i)CHGE) = \sum_{i=1}^{m+1} \frac{PS_i}{100} INSTS_0 + (m)CHGE \sum_{i=1}^{m+1} \frac{PS_i}{100}$$

$$+ CHGE \sum_{i=1}^{m+1} \frac{PS_i}{100} - CHGE \sum_{i=1}^{m+1} \frac{PS_i}{100} i$$

so

$$d_1 = x-(x-d_1) = INSTS_0 \frac{PS_m}{100} + CHGE \sum_{i=1}^{m} \frac{PS_i}{100} - CHGE \frac{PS_m}{100} = INSTS_0 \frac{PS_m}{100}$$

$$+ CHGE \sum_{i=1}^{m-1} \frac{PS_i}{100} = d_{1 \text{ for level installations}} + CHGE \sum_{i=1}^{m-1} \frac{PS_i}{100}$$

similarly

$$d_2 = INSTS_0 \frac{PS_{m+1}}{100} + CHGE \sum_{i=1}^{m} \frac{PS_i}{100} = d_{2 \text{ for level installations}} + CHGE \sum_{i=1}^{m} \frac{PS_i}{100}$$